Network Creation Games

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DISCO Seminar – FS 2011

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Matús Mihalák and Jan Christoph Schlegel.
The price of anarchy in network creation games is (mostly) constant.
In *Proceeding of the Third International Symposium on Algorithmic Game Theory, (SAGT)*, pages 276–287. Springer, 2010.

Noga Alon, Erik D. Demaine, MohammadTaghi Hajiaghayi, and Tom Leighton.
Basic network creation games.
In *Proceedings of the 22nd ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 106–113, New York, NY, USA, 2010. ACM.

The Game

A. Fabrikant, A. Luthra, E. Maneva, C. H. Papadimitriou, S. Shenker, PODC '03

- Creation and maintenance of a network is modeled as a game
- n players – vertices in an undirected graph
- can buy edges to other players for a fix price $\alpha > 0$ per edge
- The goal of the players: minimize a cost function:

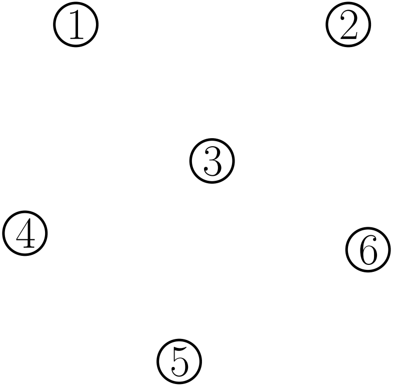
$$cost_u = creation\ cost + usage\ cost$$

The Game

$$cost_u = creation\ cost + usage\ cost$$

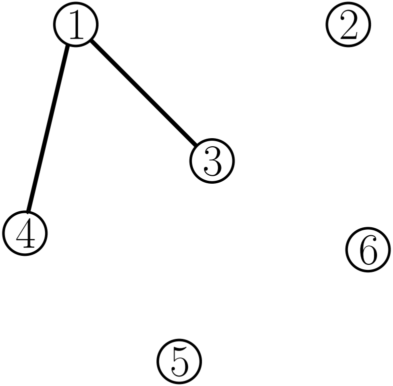
- creation cost: $\alpha \cdot$(number of edges player u buys)
- usage cost for player u:
  - SumGame (Fabrikant et al. PODC 2003)
    Sum over all distances $\sum_{v \in V} d(u, v)$
    average-case approach to the usage cost
  - MaxGame (Demaine et al. PODC 2007)
    Maximum over all distances $\max_{v \in V} d(u, v)$
    worst-case approach to the usage cost

Example



$$s_1 = \{3, 4\}$$
$$s_2 = \{1, 3\}$$
$$s_3 = \{5\}$$
$$s_4 = \{3\}$$
$$s_5 = \{\}$$
$$s_6 = \{3\}$$

Example



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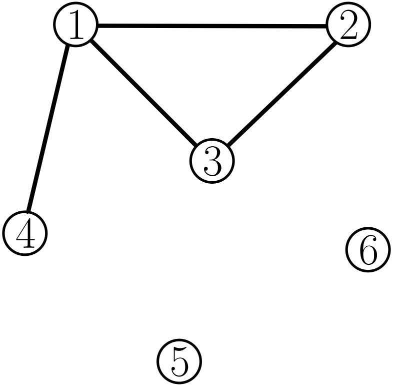

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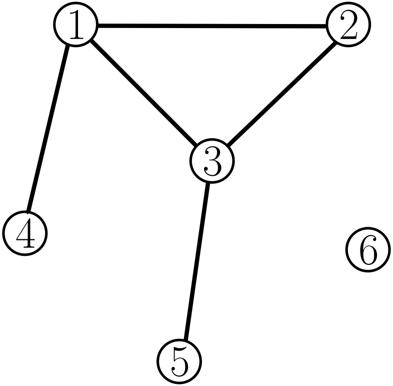

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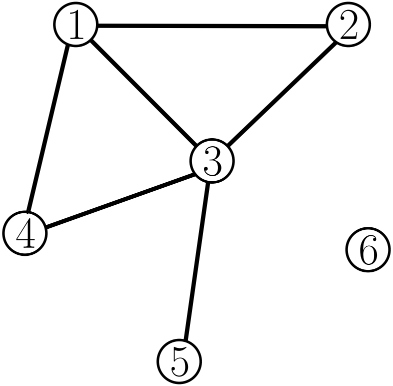

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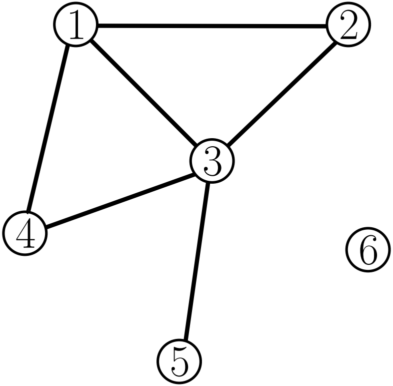

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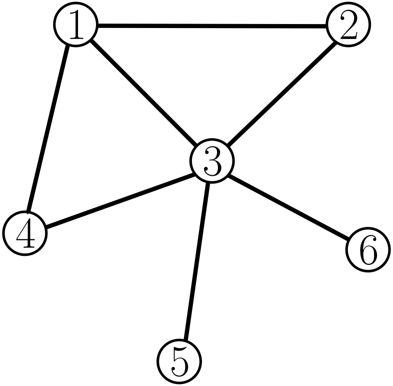

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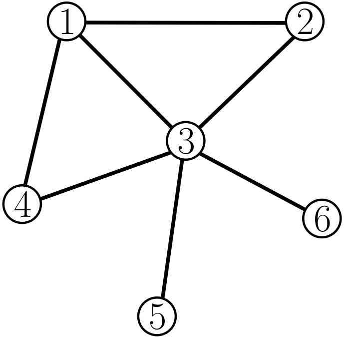
Example

SumGame:



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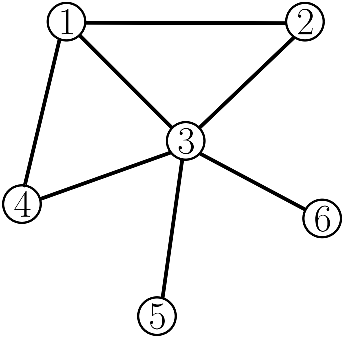
$$cost_1 = 2\alpha + 1 + 1 + 1 + 2 + 2 = 2\alpha + 7$$

etc.

Example

MaxGame:



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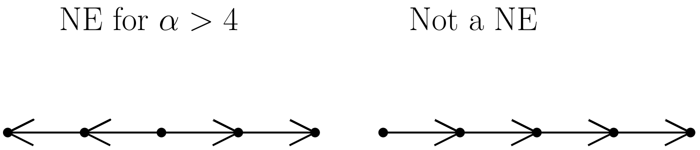
$cost_1 = 2\alpha + 2$
etc.

Nash Equilibrium

We consider Nash equilibria, i.e. graphs where no player can improve by deleting some of her/his edges and/or buying new edges Simple example:



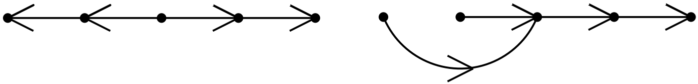
The arrows indicate who bought the edges (point from buyer away)

Nash Equilibrium

We consider Nash equilibria, i.e. graphs where no player can improve by deleting some of her/his edges and/or buying new edges Simple example:

NE for $\alpha > 4$

Not a NE



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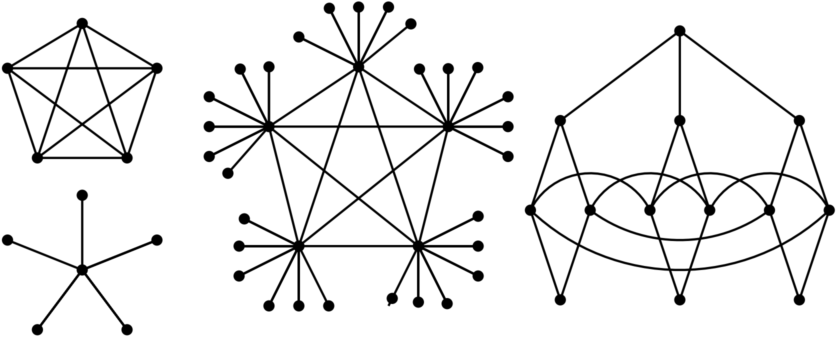
More examples



Nash Equilibria (for appropriate choice of α and of strategy profiles)

Price of Anarchy

We are interested in large networks: Typical questions:

- What network topologies are formed? What families of equilibrium graphs can one construct for a given α?
- How efficient are they? Price of Anarchy

$$\text{PoA} = \frac{\textit{Cost}(\text{worst-case equilibrium})}{\textit{Cost}(\text{social optimum})}.$$

- constant PoA $\rightsquigarrow$ equilibrium networks efficient

Previous Results

- (Fabrikant et al. PODC 2003) Definition of the game, $PoA = \mathcal{O}(\sqrt{\alpha})$ in SumGame, The PoA is bounded by the diameter for most α
- (Albers et al. SODA 2006) The PoA in SumGame is constant for $\alpha = \mathcal{O}(\sqrt{n})$ and $\alpha \geq 12n \log n$, Improved general bound
- (Demaine et al. PODC 2007) The PoA is constant for $\alpha < n^{1-\varepsilon}$, first $o(n^{\varepsilon})$ general bound, Introduction of MaxGame, Several bounds for the PoA in MaxGame

Previous Results

MaxGame:

$\alpha =$	0 – $2\sqrt{\log n}$	$2\sqrt{\log n}$ – n	n – ∞
previous	$\mathcal{O}(n^{2/\alpha})$	$\mathcal{O}(\min\{4^{\sqrt{\log n}}, (n/\alpha)^{1/3}\})$	≤ 2

SumGame:

$\alpha =$	0 – 1	1 – 2	2 – $\sqrt[3]{n/2}$	$\sqrt[3]{n/2}$ – $\sqrt{n/2}$	$\sqrt{n/2}$ – $O(n^{1-\epsilon})$	$O(n^{1-\epsilon})$ – $12n\lg n$	$12n\lg n$ – ∞
previous	1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{\mathcal{O}(\sqrt{\log n})}$	≤ 1.5

Our Results

MaxGame:

$\alpha =$	0 – $\frac{1}{n-2}$	$\frac{1}{n-2}$ – $\mathcal{O}(n^{-\frac{1}{2}})$	$\mathcal{O}(n^{-\frac{1}{2}})$ – 129	129 – $2\sqrt{\log n}$	$2\sqrt{\log n}$ – n	n – ∞
new	1	$\Theta(1)$	$2^{\mathcal{O}(\sqrt{\log n})}$	≤ 4		≤ 2
previous	$\mathcal{O}(n^{2/\alpha})$				$\mathcal{O}(\min\{4^{\sqrt{\log n}}, (n/\alpha)^{1/3}\})$	≤ 2

Our Results

SumGame:

$\alpha = \quad 0 \quad 1 \quad 2 \quad \sqrt[3]{n/2} \quad \sqrt{n/2} \quad O(n^{1-\epsilon}) \quad 273n \quad 12n \lg n \quad \infty$

new	1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{\mathcal{O}(\sqrt{\log n})}$	< 5	≤ 1.5
previous	1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{\mathcal{O}(\sqrt{\log n})}$	$2^{\mathcal{O}(\sqrt{\log n})}$	≤ 1.5

Main result for SumGame

Theorem

For $\alpha > 273n$ every equilibrium graph is a tree.

As Fabrikant et al. proved that trees have $PoA < 5$ this implies:

Corollary

For $\alpha > 273n$ the price of anarchy is smaller than 5.

Up to a constant factor this is the best result one can obtain:

Proposition (Albers et al. 2006)

For $\alpha < n/2$ there are non-tree equilibrium graphs.

All equilibria are trees for $\alpha > Cn$

Some intuition why this could be true:

- Equilibrium graphs become sparser with increasing α. More precisely it is easy to show the following:

Lemma

The average degree of an equilibrium graph is $\mathcal{O}(1 + \frac{n}{1+\alpha})$.

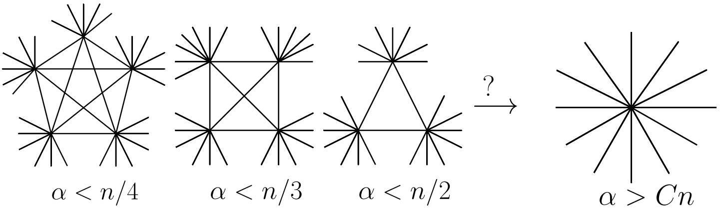
- We show a (much) stronger version of the lemma:

Lemma

Let H be a biconnected component of an equilibrium graph G for $\alpha > n$ *then for the average degree of H,* $d(H) \leq 2 + \frac{8n}{\alpha - n}$.

All equilibria are trees for $\alpha > Cn$

- Albers et al. showed that k stars of size n/k whose centers are connected to a clique is an equilibrium graph for $\alpha < n/(k-1)$:



Idea: Look at biconnected components and prove that they contain "few" vertices of the whole graph.

All equilibria are trees for $\alpha > Cn$

Lemma (1)

Let H be a biconnected component of an equilibrium graph G for $\alpha > n$ then $d(H) \leq 2 + \frac{8n}{\alpha - n}$.

Lemma (2)

Let H be a biconnected component of an equilibrium graph G for $\alpha > 19n$ then $d(H) \geq 2 + \frac{1}{34}$.

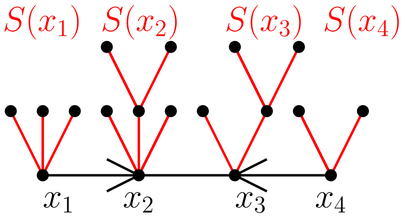
- Both proofs: look at the local structure of equilibrium graphs
- Main difficulty: it matters who buys a certain edge in the graph!

Proof Idea

Lemma (2)

Let H be a biconnected component of an equilibrium graph G for $\alpha > 19n$ *then* $d(H) \geq 2 + \frac{1}{34}$.

- Show: every vertex in H has a vertex with degree 3 in H nearby
- Several cases – a simple case:



edges in H = black, edges in $V \setminus H$ = red

Assign every vertex to closest vertex in $H \rightsquigarrow S(x_i)$

Proof Idea

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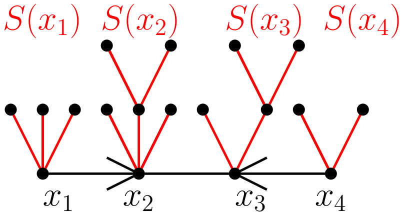

$|S(x_2)| > |S(x_3)|$

either x_1 or x_4 can improve

Proof Idea

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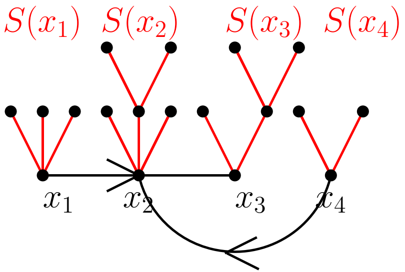

x_4 can improve by deleting x_4x_3 and buying x_4x_2...

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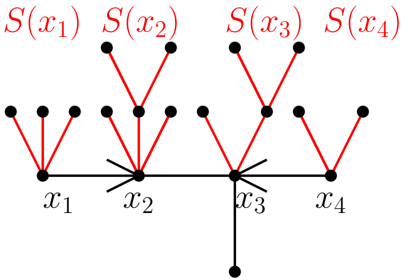

...unless x_3 has degree at least 3 in H

Put everything together:

For a biconnected component H in an equilibrium graph G:

Lemma 1:

$$\mathrm{d}(H) \leq 2 + \frac{8n}{\alpha - n}$$

Lemma 2:

$$\mathrm{d}(H) \geq 2 + \frac{1}{34}$$

The inequalities become contradicting for $\alpha > 273n$ hence:

Theorem

For $\alpha > 273n$ every equilibrium graph is a tree.

Summary

- We obtain constant bound on the PoA for most edge prices
- Still no tight bound for $\alpha = \Theta(n)$ in SumGame, $\alpha = \Theta(1)$ in MaxGame
- Interesting range occurs around the threshold for trees
- Problem with Nash equilibrium:
  - computationally intractable
  - calculating best-response NP-hard for both variants

Basic network creation games

N. Alon, E. D. Demaine, M. Hajiaghayi, T. Leighton, SPAA '10

Goals

- Computationally feasible solution concept
- Find "simplest and the heart of all such games"
  - reduce number of parameters, by avoiding α
  - results should generalize to previous models

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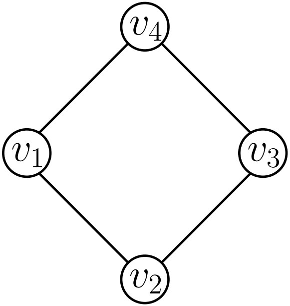
Model

- graph G is given
- players/nodes are only allowed to "swap": Delete an adjacent edge and build a new one instead
- G is in swap equilibrium if no player u can swap one edge and improve its usage cost $\sum_{v \in V} d_G(u, v)$

Example



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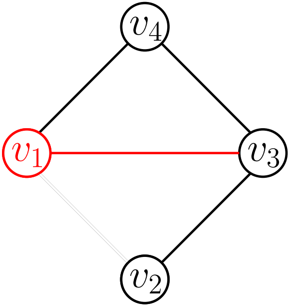

v_1 swaps v_1v_2 with v_1v_3

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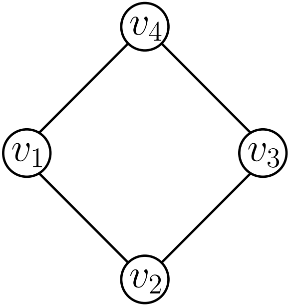

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- v_1 does not improve from swapping: $\text{new usage cost} = 4 = \text{old usage cost}$

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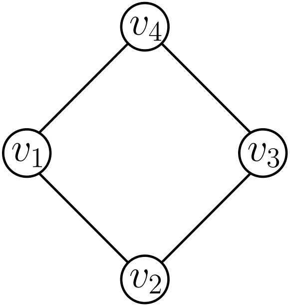

- v_1 does not improve from swapping: $\text{new usage cost} = 4 = \text{old usage cost}$
- by symmetry also v_2, v_3, v_4 cannot improve from swapping

$\Rightarrow$ the 4-cycle is a swap equilibrium

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- Computationally feasible solution concept
  - best response can be calculated in poly time:
    $\mathcal{O}(n^2)$ possible swaps
    Calculating usage cost via BFS-search: $\mathcal{O}(n^2)$
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    $\exists$ Nash equil. which are not swap equil. and vice versa

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- Find "simplest and the heart of all such games" ?
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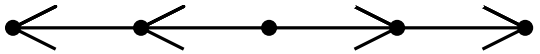
    $\exists$ Nash equil. which are not swap equil. and vice versa

Nash vs. Swap

Difference:

- original game: only the player who bought an edge can swap!
- basic network creation game: both ends of an edge can swap!

NE for $\alpha > 4$



Not a Swap Equil.

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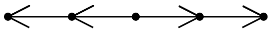

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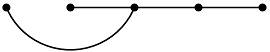

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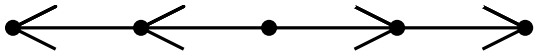

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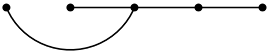

Proposal for a modification:

- add an orientation to the graph indicating who owns an edge
- players are only allowed to swap edges that they own
- G is in directed swap equilibrium if no player can swap an edge which he/she owns and improve

Directed Basic Network Creation Game

Advantage:

- Best response can still be calculated in poly time
- This generalizes both Nash equilibrium and swap equilibrium

$$\boxed{\textsc{Swap} \cup \textsc{Nash} \subset \textsc{DirectedSwap}}$$

Directed Basic Network Creation Game

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- Best response can still be calculated in poly time
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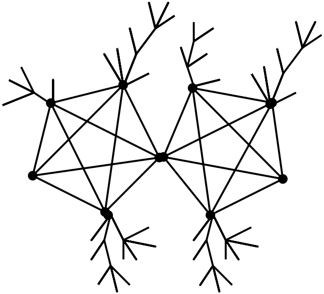
Problem:

- Proofs become more technical than in the (undirected) Basic Network Creation Game
- Anyhow we can prove some interesting things:

Structure of equilibrium graphs

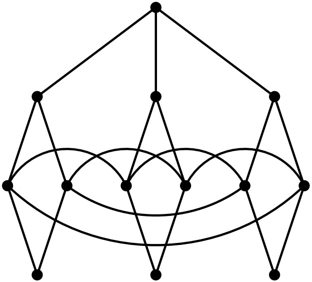
Theorem

Every equilibrium graph has at most one 2-edge-connected component.

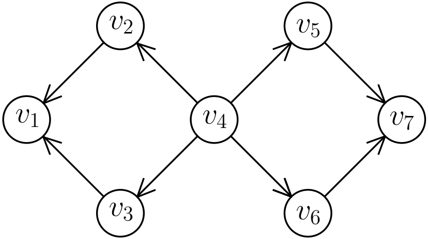


- This holds for the various equilibrium concepts: Nash, swap, directed swap
- Equilibrium graphs are bridgeless graphs "with trees attached"
- The attached trees have diameter $\mathcal{O}(\log n)$
- Bounds for the diameter of 2-edge-connected component?

Lower bounds for bridgeless graphs



(a) Diameter-3 swap equilibrium (Alon et al.)



(b) Diameter-4 directed swap equilibrium

Upper bounds

Conjecture

The diameter of an equilibrium graph is $\mathcal{O}(\log n)$.

- For Nash and Directed Swap we have matching lower bound
- under strong assumption on the degree distribution we can prove logarithmic upper bound:

Theorem

If the unique 2-edge connected component H *has minimum degree* $d(H) \geq n^{\varepsilon}$ *for* $0 < \varepsilon < 1$ *then there is a constant* $C(\varepsilon) > 0$ *depending on* ε *such that* $diam(H) \leq C(\varepsilon)$.

- Best general upper bound: $\mathcal{O}(2^{\sqrt{\log n}})$

Open problems

- What is the "right" model? Original vs. Basic vs. Directed Basic
- What other bridgeless equilibria can we construct? Can we achieve non-constant diameter?
- Can you prove a logarithmic bound on the diameter in any of those models?
- Make the model dynamic

Thank you for your attention! Questions?