The Round Complexity of Distributed Sorting

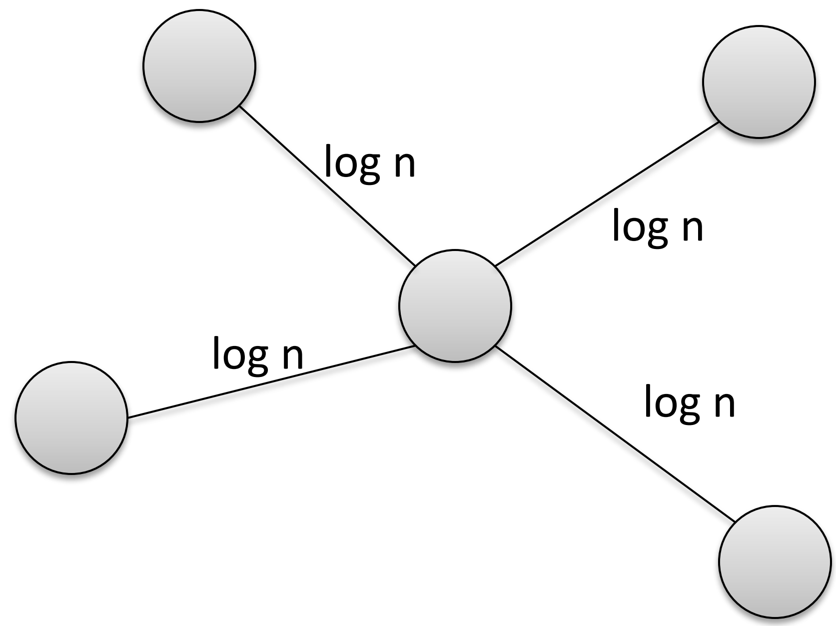
by
Boaz Patt-Shamir
Marat Teplitsky
Tel Aviv University

Motivation

- Distributed sorting
- Infrastructure is more and more distributed
  - Cloud
  - Smartphones

CONGEST model

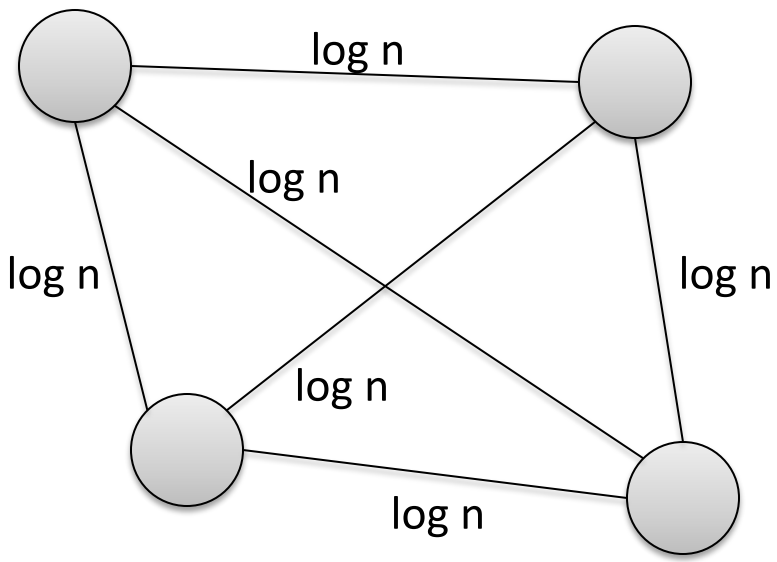
- Models congestion in a network
  - Bandwidth restriction: Message complexity in O(log n)
- Abstract model
  - Removes complexity



Network

- Fully connected network (clique)
- Message in O(log n)
- Synchronous rounds

Switch with 512 ports



Problem statement

- Number of nodes: *n*
  - Denoted as $V = \{v_1,\ldots,v_n\}$
- Input Values: max *n* per node
  - Max n^2 in total
- Goal: Sort in O(log log n) rounds w.h.p

- Definition
  - With high probability (w.h.p): $1\text{-}n^{-O(1)}$

The algorithm – Overview

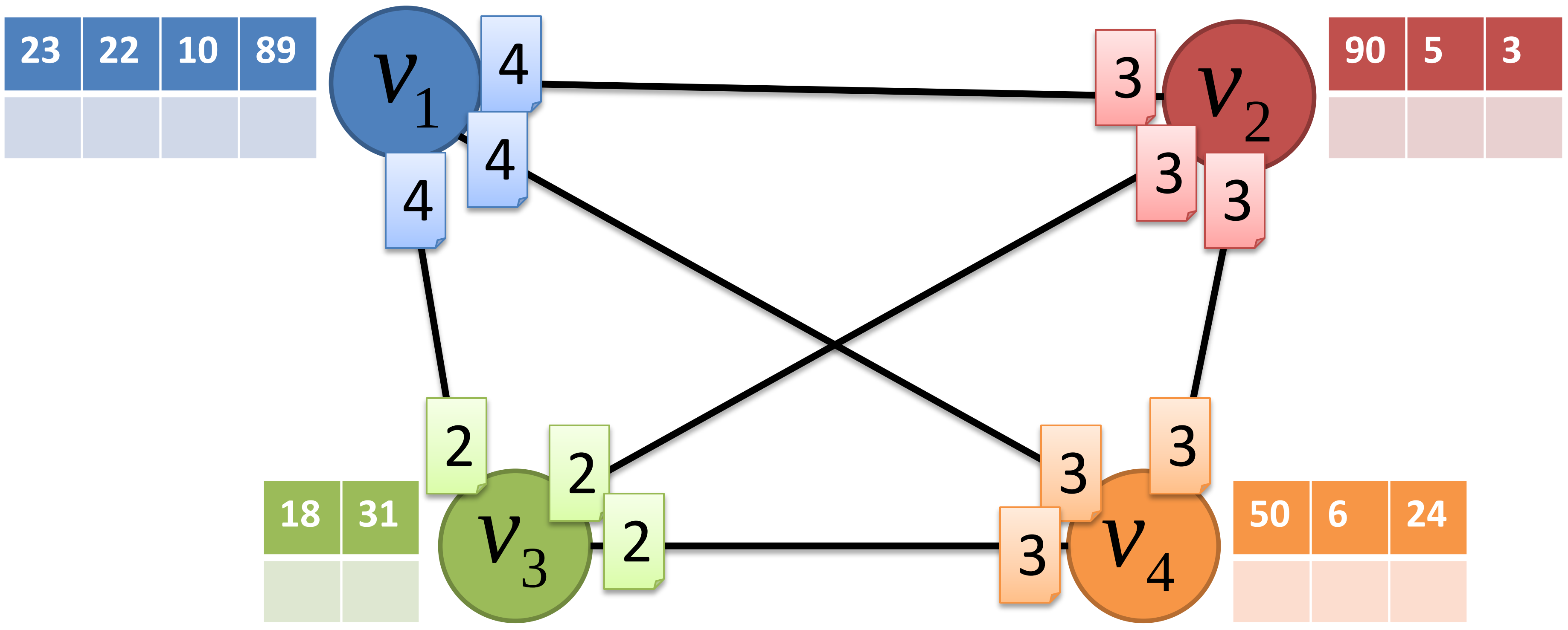
- Split the input values into *n* ranges
- Each node sorts one range
  - Send input values to the corresponding node

The algorithm – Partition phase

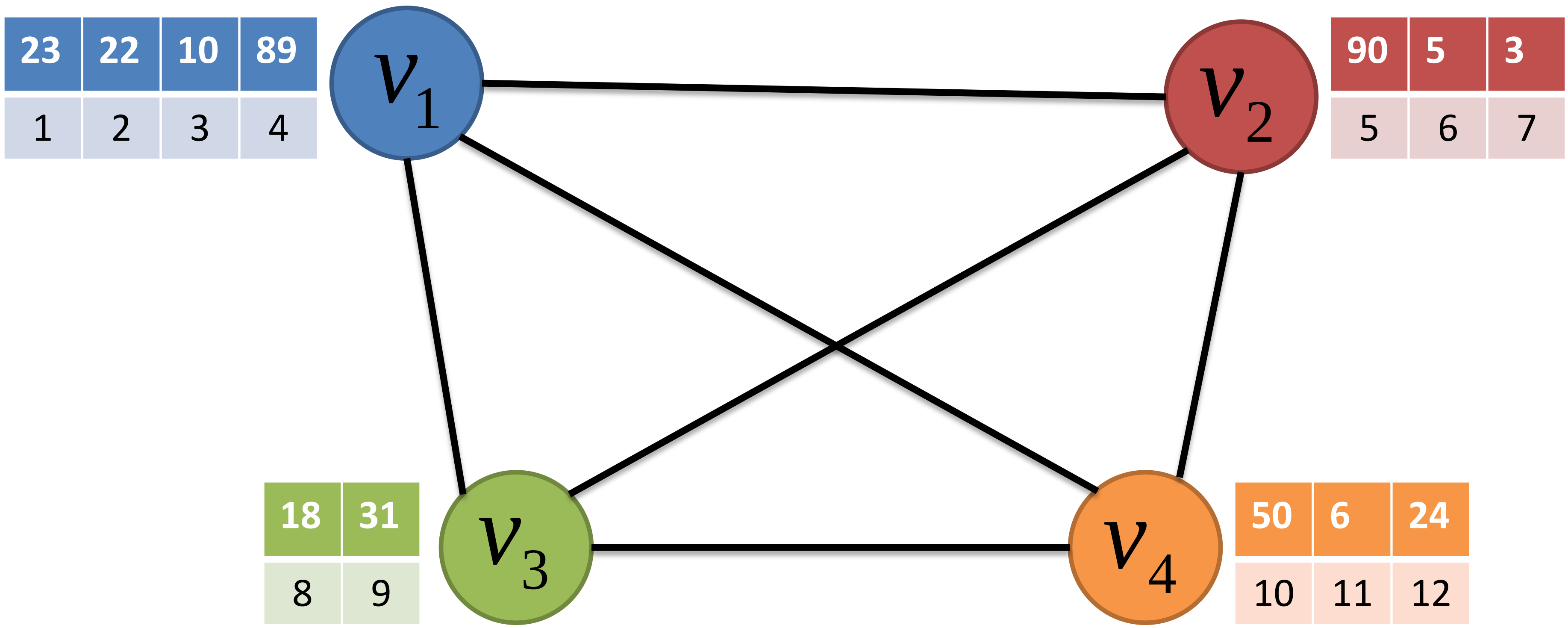
- Create a global order on the keys
  - The nodes are order by their id
  - Each node creates an arbitrarily local order
  - The global order is then $\sum_{l=1}^{i-1} a_i + k_{ij}$

- Partition them into *n* disjoint ranges

The algorithm – Partition phase

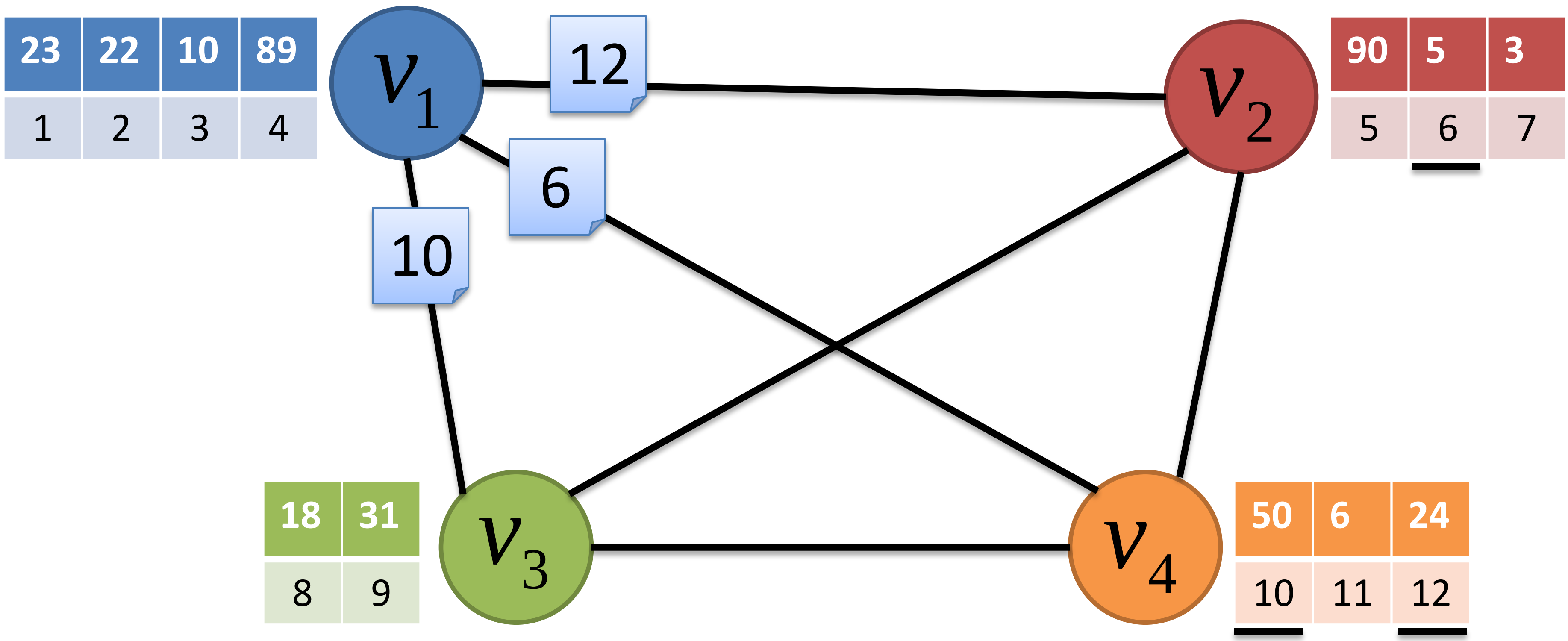


The algorithm – Partition phase



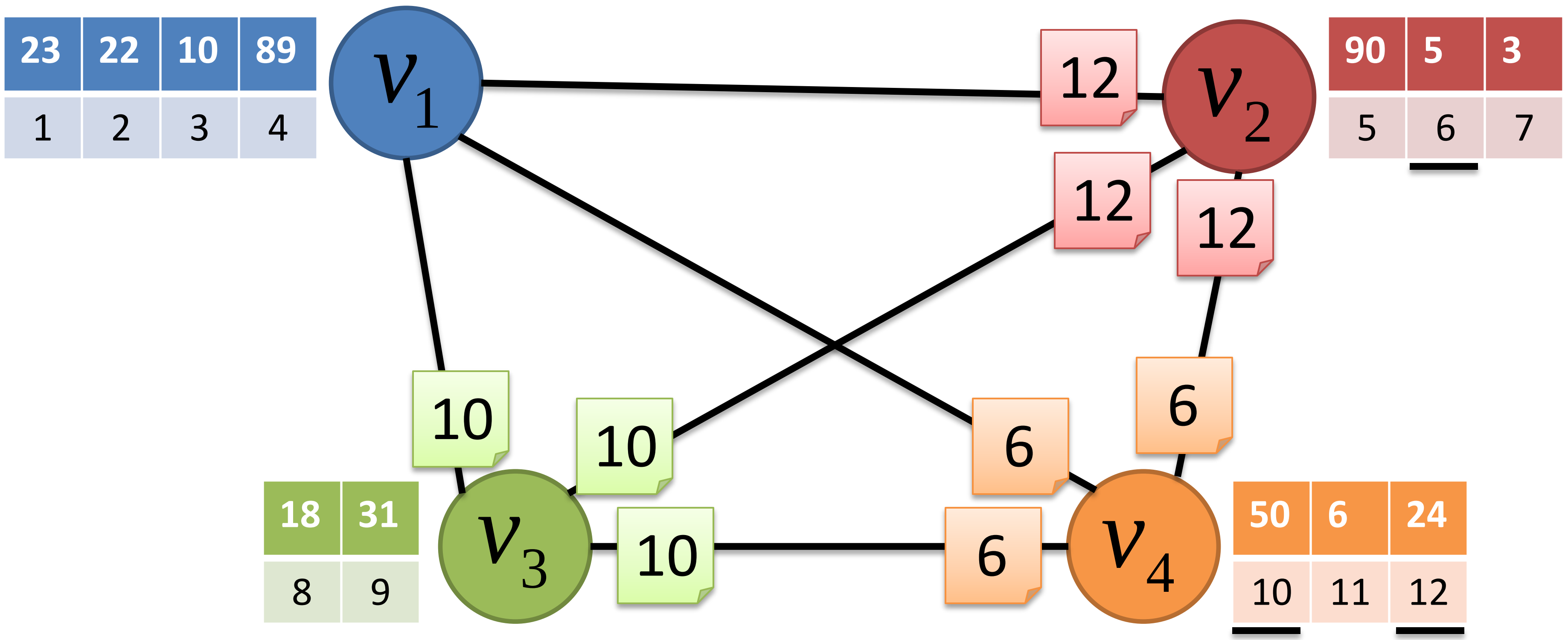
The algorithm – Partition phase

Choose order of delimiters: 12,6,10



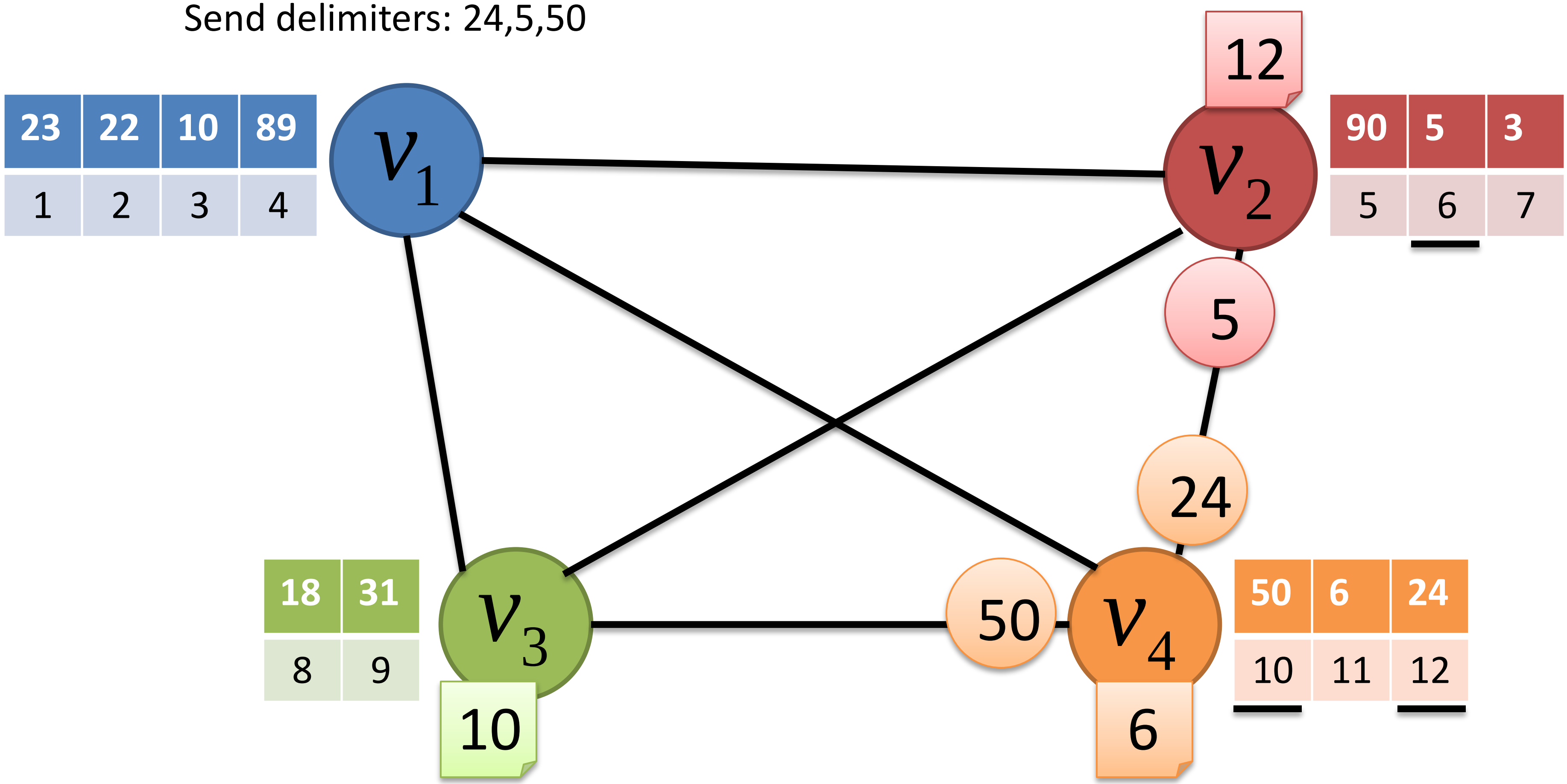
The algorithm – Partition phase

Broadcast order of delimiters: 12,6,10



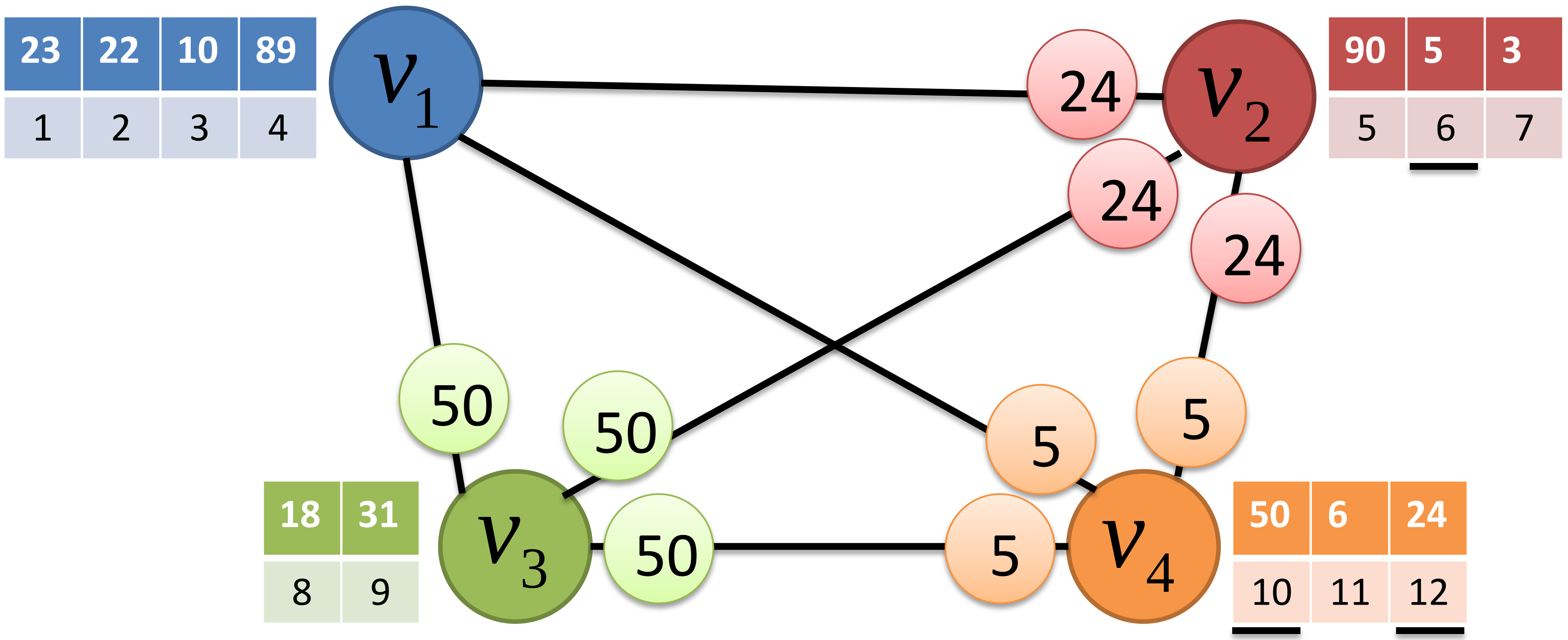
The algorithm – Partition phase

Send delimiters: 24,5,50

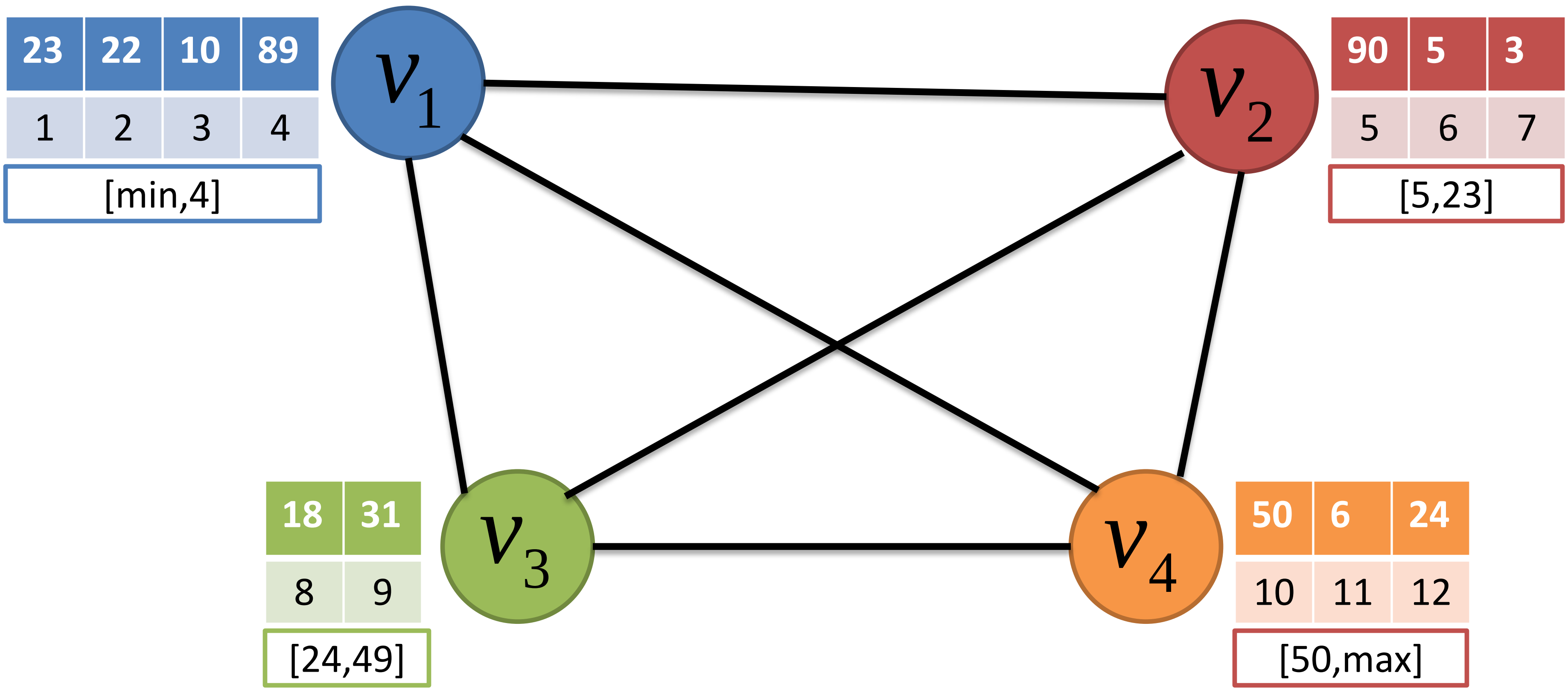


The algorithm – Partition phase

Broadcast delimiters: 24,5,50



The algorithm – Partition phase



The algorithm – First stage

- Only nodes with max *2n ln ln n* keys in their range are *active* nodes in this phase
- Keys are *active* if their destination node is active.

The algorithm – First stage

repeat

- for each active key pick intermediate destination (source node)
- for each final destination, pick one key and send it (intermediate node)
- Send all other received keys back

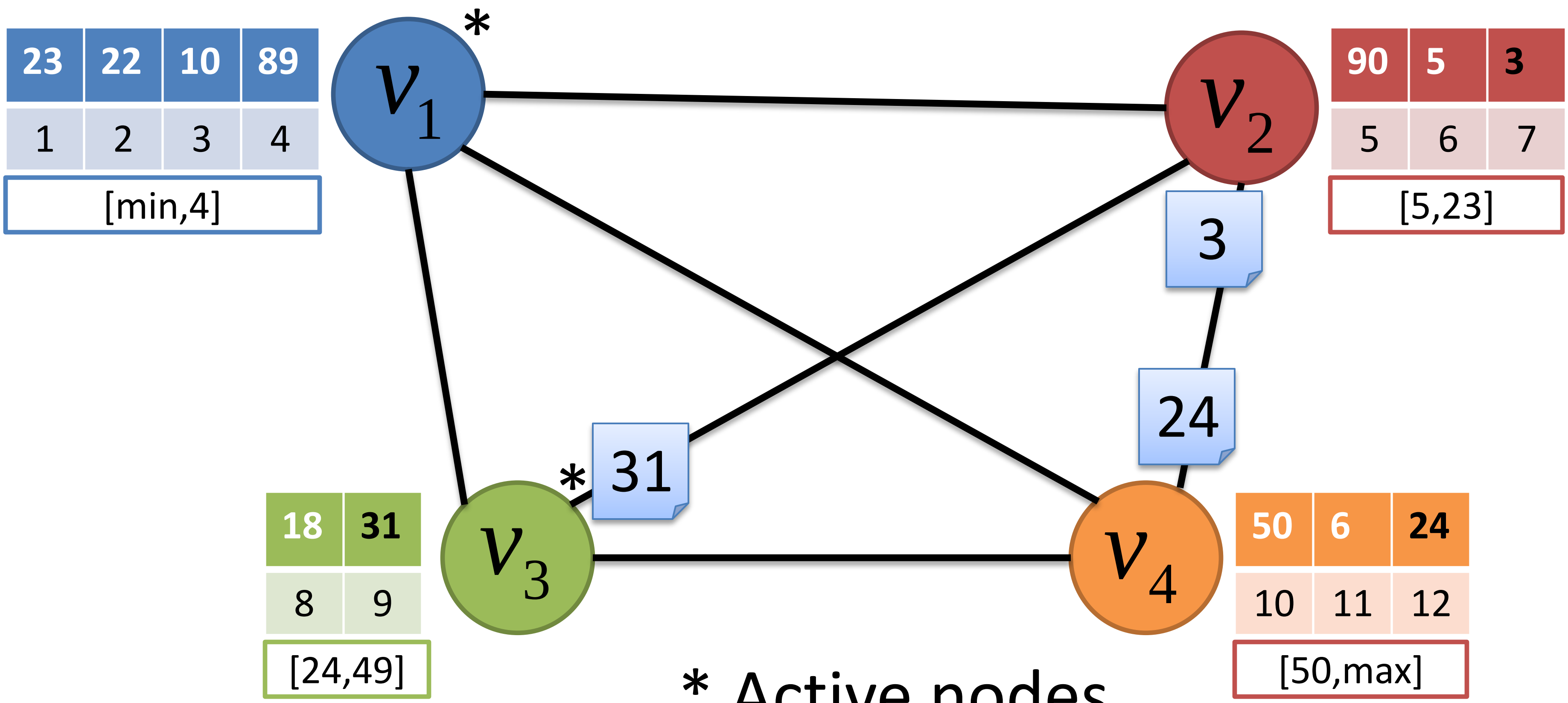
until all active key reached their destination

The algorithm – First stage

- Active nodes: Max 2 keys in the range

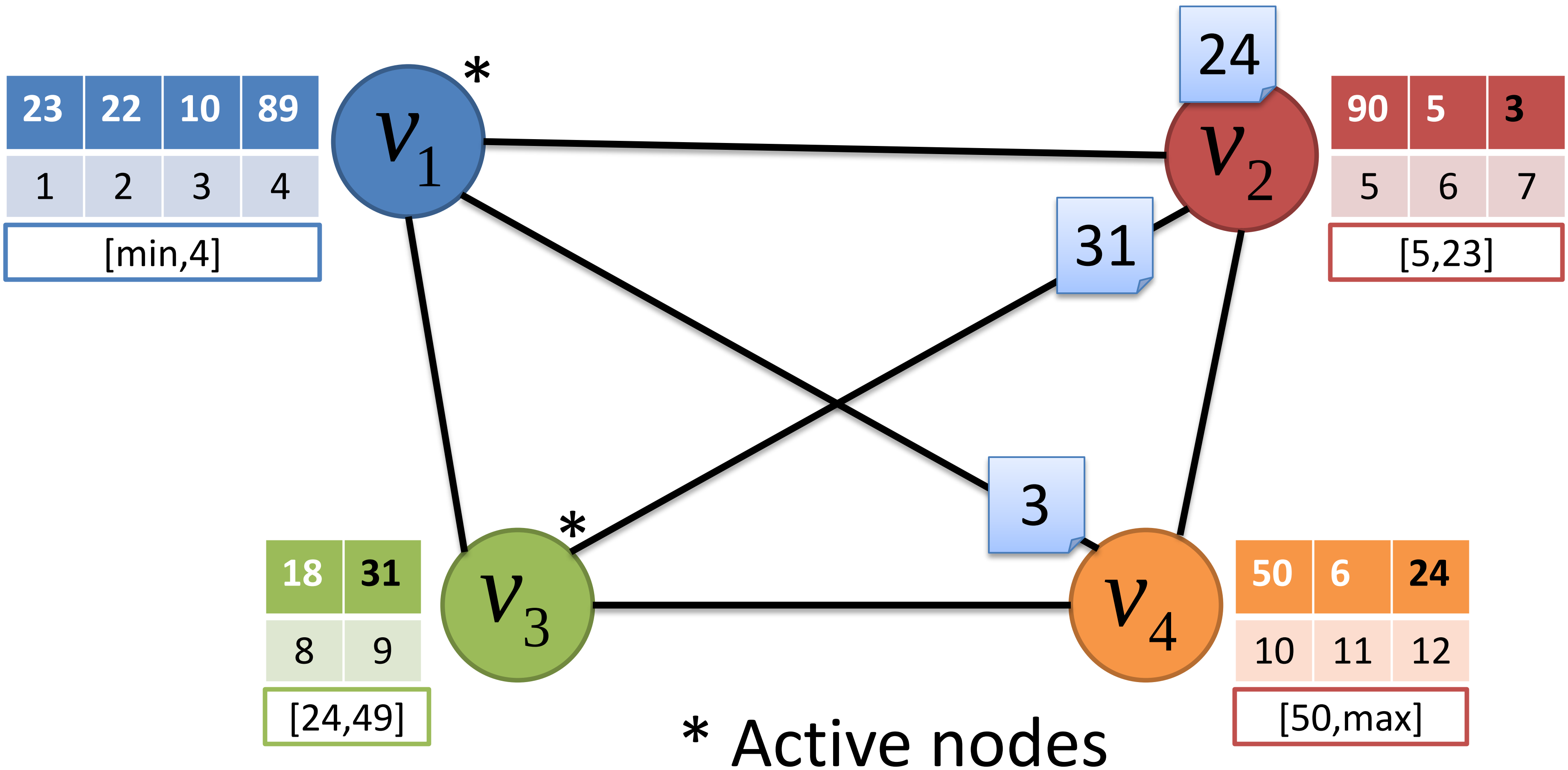
V_1	V_2	V_3	V_4
[min,4]	[5,23]	[24,49]	[50,max]
3	5, 6, 10, 18, 23	24, 31	50, 89, 90

The algorithm – First stage

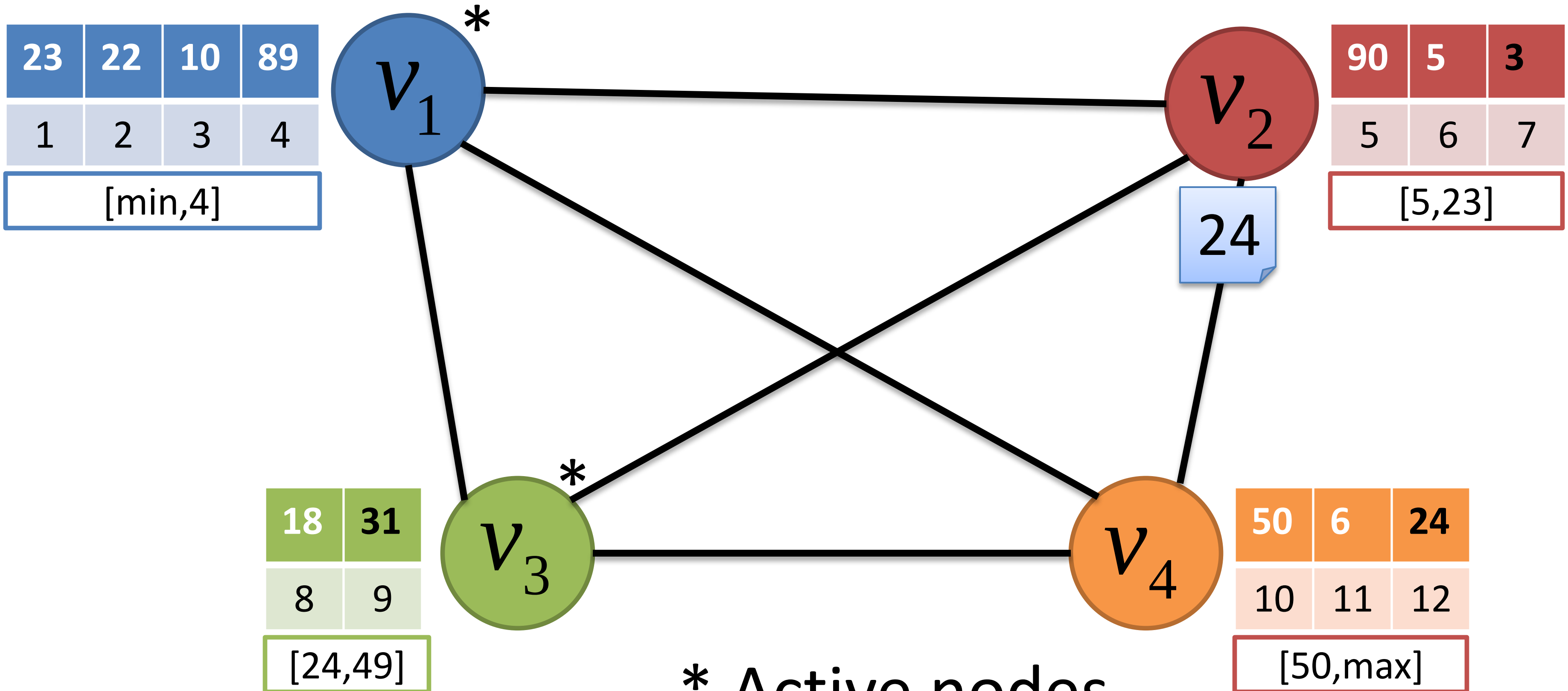


* Active nodes

The algorithm – First stage

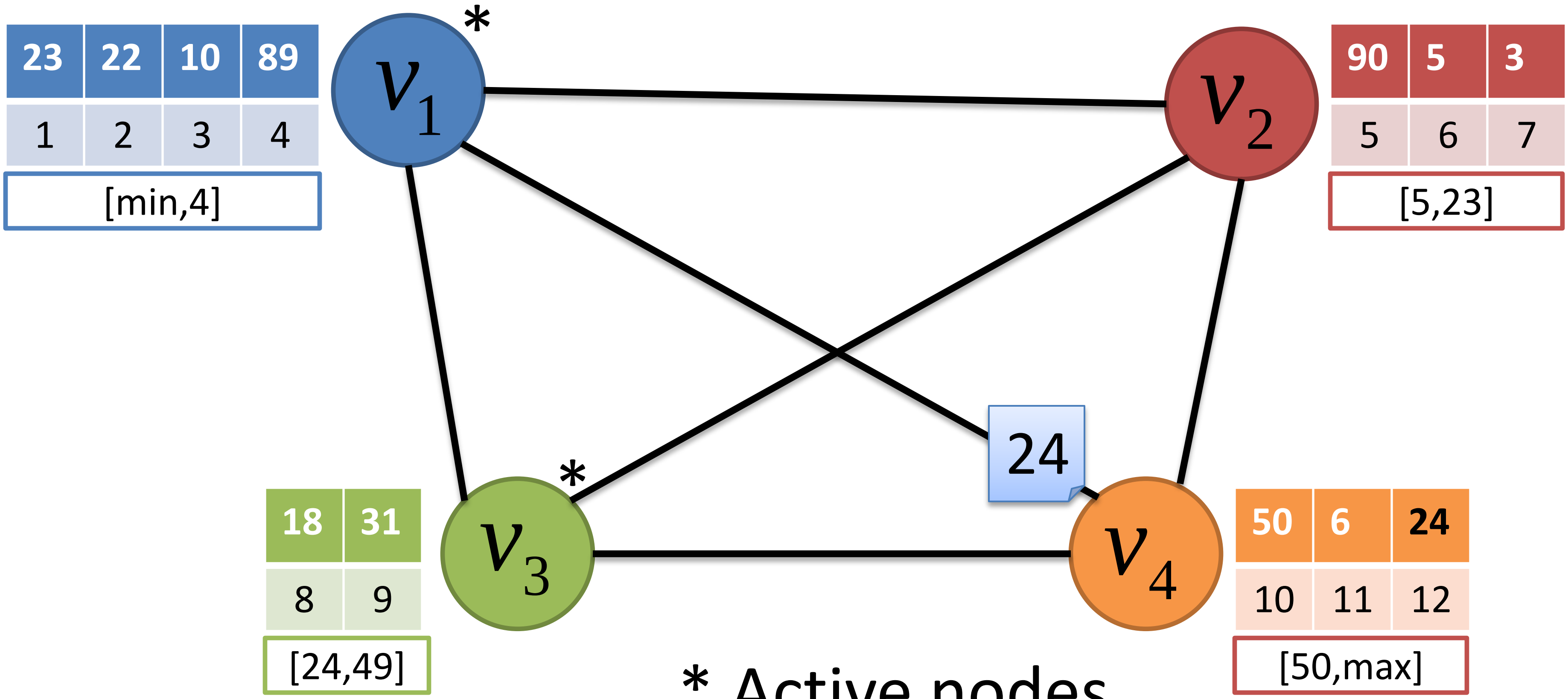


The algorithm – First stage



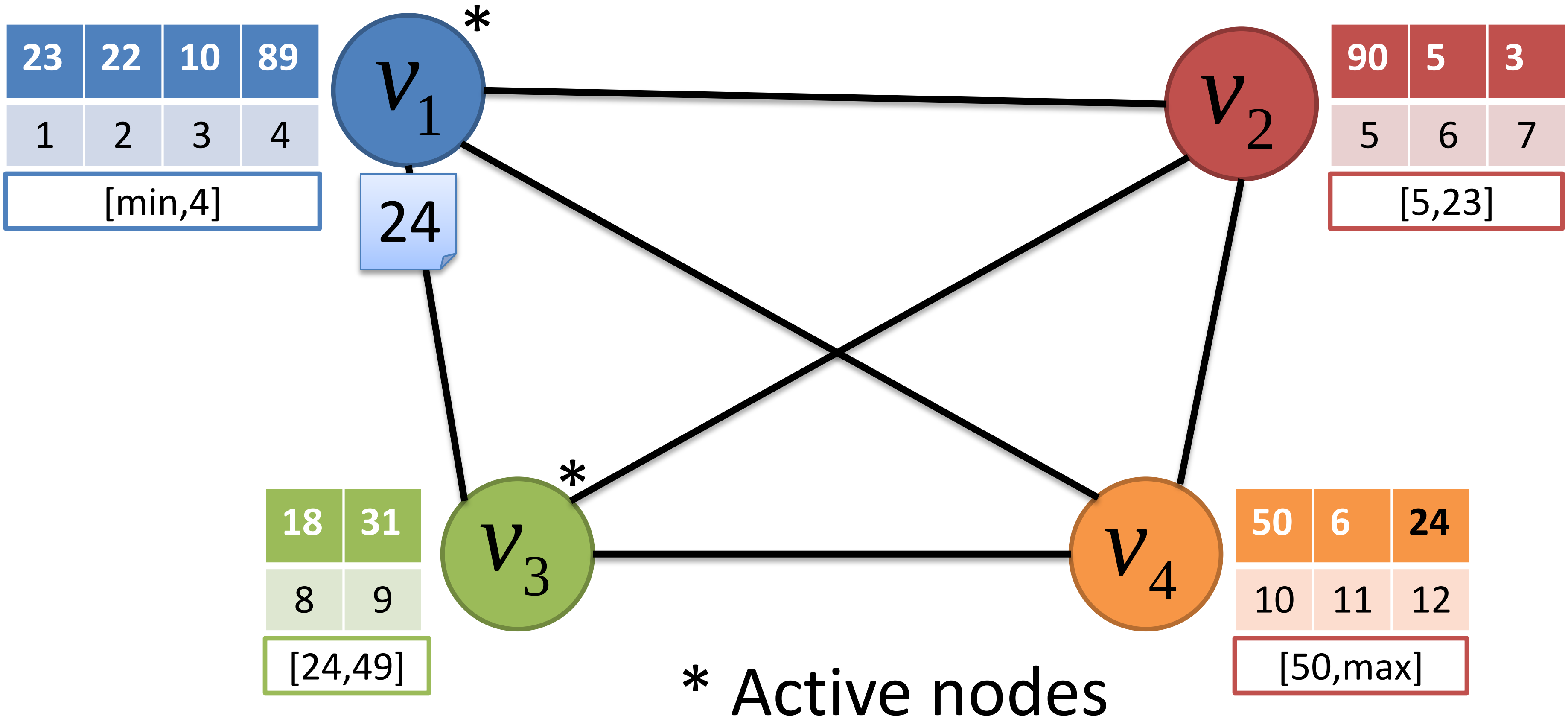
* Active nodes

The algorithm – First stage



* Active nodes

The algorithm – First stage



The algorithm – Cleanup stage

- Do the same for the nodes with more than $2\ n\ ln\ ln\ n$ keys.

- Local sort the keys

Chernoff bound

Analysis

- Lemma 1.1:
  - With high probability, the number of non-selected segments is at most $\frac{2n}{\ln n \ln \ln n}$.

Analysis

- Lemma 1.2:
  - With high probability, the number of ranges with more than *2n ln ln n* keys is at most $\frac{2n}{\ln n \ln \ln n}$.

Analysis

- Lemma 2.1:
  - W.h.p., the number of keys remaining to the cleanup stage is at most $\frac{4n^2}{\ln n}$.

Analysis

- Lemma 2.2:
  - In the cleanup stage, w.h.p., all ranges are of size O(n).

Analysis

- Lemma 3.1:
  - If there are **more** than *n* active keys with destination v_i, then w.h.p. at least $\frac{n}{9}$ keys will be delivered at v_i in one iteration.

Analysis

- Lemma 3.2:
  - If there are **at most** *n* active keys with destination v_i, then w.h.p. all keys will be delivered in $O(\ln \ln n)$ iterations.

Related Work

- Concurrently to this paper, Lenzen and Wattenhofer proved the following:
  - Suppose there are O(n) messages in each node and the number of messages destined to each node is O(n), then routing all messages can be done in O(1)
- With this, the algorithm can be improved to work in O(1)

Q&A