Crash course – Verification of Finite Automata
Binary Decision Diagrams

Exercise session - 24.11.201
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Equivalence of representations

Sets
- Set algebra
- $\cup, \cap, \neg$

Boolean functions/Characteristic functions
- Boole algebra
- $+, \cdot, \neg$

$\psi_E = 1$
$\psi_A = f$
$\psi_B = g$
$\psi_{A \cap B} = f \cdot g$
Equivalence of representations

Sets

- Set algebra
- $\cup, \cap, \neg$

Boolean functions/Characteristic functions

- Boole algebra
- $+, \cdot, \neg$

Binary encoding:

$\sigma: X \rightarrow \Psi$

$\Rightarrow$ Map a state to a binary expression

$\Rightarrow$ Bijective mapping!
Equivalence of representations

- Set algebra
- \( \cup, \cap, \neg \)

Binary encoding:
\[
\sigma: X \rightarrow \Psi
\]
- Map a state to a binary expression
- Bijective mapping!
Equivalence of representations

- **Sets**
  - \( A \)
  - \( s \in A \)

- **Example:**
  \[ \sigma(s) = x_1 \overline{x_0} = (1,0) \]
  \[ \psi_A = x_1 + x_0 \]
  \[ \rightarrow s \models \psi_A \]

- **Boolean functions/Characteristic functions**
  - \( \psi_E = 1 \)
  - \( \psi_A = f \)
  - \( \psi_B = g \)
  - \( \psi_{A \cap B} = f \cdot g \)

  \[ \sigma(s) \models \psi_A \]
  or just \( s \models \psi_A \)

  Reads “s satisfies \( \psi_A \)”
Binary Decision Diagrams

Based on the Boole-Shannon decomposition:

\[
f = \overline{x} \cdot f \bigg|_{x=0} + x \cdot f \bigg|_{x=1}
\]

Boolean function of \( n \) and \((n - 1)\) variables

→ For a given order of variable, the decomposition is unique!
→ Hence the uniqueness of O(rdered)BDD.

Reminder:
In practice, simplicity of BDD depends strongly on the order.

Good vs. Bad ordering
Binary Decision Diagrams: an example

\[ f: x_1 + \overline{x}_1 x_2 + \overline{x}_2 x_3 \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} x_2 + \overline{x_2} \overline{x_3} \]

Fall \( x_1 = 0 \)

\[ f_{|x_1=0} : x_2 + \overline{x_2} \overline{x_3} \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]

Fall \( x_1 = 0 \)

\[ f_{|x_1=0} : x_2 + \overline{x_2} \overline{x_3} \]

Fall \( x_2 = 0 \)

\[ f_{|x_1=0,x_2=0} : \overline{x_3} \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]

Fall \( x_1 = 0 \)

\[ f|_{x_1 = 0} : x_2 + \overline{x_2} x_3 \]

Fall \( x_2 = 0 \)

\[ f|_{x_1 = 0, x_2 = 0} : \overline{x_3} \]
Binary Decision Diagrams: an example

\[ f: x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]

Fall \( x_1 = 0 \)
\[ f|_{x_1=0}: x_2 + \overline{x_2} x_3 \]
Fall \( x_2 = 0 \)
\[ f|_{x_1=0, x_2=0}: \overline{x_3} \]
Fall \( x_2 = 1 \)
\[ f|_{x_1=0, x_2=1}: 1 \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x}_1 x_2 + \overline{x}_2 x_3 \]

Fall \( x_1 = 0 \)
\[ f_{|x_1=0} : x_2 + \overline{x}_2 x_3 \]
  Fall \( x_2 = 0 \)
\[ f_{|x_1=0,x_2=0} : \overline{x}_3 \]
  Fall \( x_2 = 1 \)
\[ f_{|x_1=0,x_2=1} : 1 \]

Fall \( x_1 = 1 \)
\[ f_{|x_1=1} : 1 \]
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Binary Decision Diagrams

Your turn!

Slides online on my webpage:
http://people.ee.ethz.ch/~jacobr/
Ex1: Sets Representation

“Each state is either a nominal or an error state”.

\[ N \cup E = X \iff \psi_N + \psi_E = 1 \]
Ex1: Sets Representation

“If a state is in the overflow set, it is not a nominal state”.

⇒ \[ N \cap O = \emptyset \iff \psi_N \cdot \psi_O = 0 \]

But note it is not necessarily true !!
Although you would like it to be...
Ex1: Sets Representation

Describe $Q_1$, the set of error states which are not an overflow, in term of sets and characteristic functions.

\[ Q_1 = E \setminus O \iff \psi_{Q_1} = \psi_E \cdot \overline{\psi_O} \]
Ex1: Sets Representation

- Describe Q2, satisfying "O ⇒ E", i.e., the set of state for which this property holds, in term of sets and characteristic functions.

Is that correct?
Ex1: Sets Representation

- Describe Q2, satisfying "O ⇒ E", i.e., the set of state for which this property holds, in term of sets and characteristic functions.

Is that correct? No!

→ What if a state is not in O?
   Property is always true!
Describe Q2, satisfying "O ⇒ E", i.e., the set of state for which this property holds, in term of sets and characteristic functions.

Is that correct? No!

→ What if a state is not in O?

Property is always true!

\[ Q_2 = (O \cap E) \cup \overline{O} = (O \cup \overline{O}) \cap (E \cup \overline{O}) = X \cap (E \cup \overline{O}) = E \cup \overline{O} \iff \psi_{Q_2} = \psi_E + \overline{\psi_O} \]
Ex2.1 Verication using BDDs

\[ a) \quad f_2 : y = \overline{x_1 + x_2 + x_3} + \overline{x_1 + x_2 + x_3} + \overline{x_1 + x_2 + x_3} \]
Ex2.1 Verification using BDDs

\[ f_1 : (x_1 \overline{x_2} + x_1 x_3 + \overline{x_2} x_3 + \overline{x_1} x_2 \overline{x_3}) \]

**Fall** \( x_1 = 0 \)

\[ y|x_1=0 = \overline{x_2} x_3 + x_2 \overline{x_3} \]

**Fall** \( x_2 = 0 \)

\[ y|x_1=0,x_2=0 = x_3 \]

**Fall** \( x_2 = 1 \)

\[ y|x_1=0,x_2=1 = \overline{x_3} \]

**Fall** \( x_1 = 1 \)

\[ y|x_1=1 = \overline{x_2} + x_3 + \overline{x_2} x_3 \]

**Fall** \( x_2 = 0 \)

\[ y|x_1=1,x_2=0 = 1 \]

**Fall** \( x_2 = 1 \)

\[ y|x_1=1,x_2=1 = x_3 \]
Ex2.1 Verication using BDDs

\[ f_2 : y = \overline{x_1 + x_2 + x_3} + \overline{x_1 + x_2 + x_3} + \overline{x_1 + x_2 + x_3} \]

Fall \( x_1 = 0 \)
\[ y|_{x_1=0} = \overline{x_2 + x_3 + x_2 + x_3} \]

Fall \( x_2 = 0 \)
\[ y|_{x_1=0, x_2=0} = x_3 + \overline{1 + x_3} = x_3 \]

Fall \( x_2 = 1 \)
\[ y|_{x_1=0, x_2=1} = \overline{1 + x_3} = x_3 \]

Fall \( x_1 = 1 \)
\[ y|_{x_1=1} = \overline{1 + \overline{1 + x_2 + x_3}} = \overline{x_2 + x_3} \]

Fall \( x_2 = 0 \)
\[ y|_{x_1=1, x_2=0} = 1 \]

Fall \( x_2 = 1 \)
\[ y|_{x_1=1, x_2=1} = x_3 \]
Ex2.2 BDDs with respect to different orderings

\[ g(x_1, x_2, y_1, y_2) = (x_1 \equiv y_1) \cdot (x_2 \equiv y_2), \quad \Pi : x_1 < x_2 < y_1 < y_2 \]

a) \[ g = x_1 \{ x_2 [y_1 (y_2) + \overline{y_1}(0)] + \overline{x_2} [y_1 (\overline{y_2}) + \overline{y_1}(0)] \} + \overline{x_1} \{ x_2 [y_1 (0) + \overline{y_1}(y_2)] + \overline{x_2} [y_1 (0) + \overline{y_1}(y_2)] \} \]

b) 

![BDD Diagram](image)
Ex2.2 BDDs with respect to different orderings

\[ g(x_1, x_2, y_1, y_2) = (x_1 \equiv y_1) \cdot (x_2 \equiv y_2) , \quad \Pi' : x_1 < y_1 < x_2 < y_2 \]

c) \[ g = x_1 \{ y_1[x_2(y_2) + \overline{x_2(y_2)}] + \overline{y_1}[0] \} \]
\[ + \overline{x_1} \{ y_1[0] + \overline{y_1}[x_2(y_2) + \overline{x_2(y_2)}] \} \]

Better ordering:
6 vs. 9 nodes
About the lecture exercise feedback
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See you next week!