Game-Theoretic Aspects of a Shirker Game

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Contents

1 Introduction

There was a time when networks were small, all operators knew each other, and there was mutual agreement on acceptable behaviour as well as the capability to enforce that. These times being long past, studying the performance of algorithms under various failure models, as well as developing algorithms resilient against some failures, has been an important area of research. In the usual topology of failure models, we find various kinds of hardware defects (*Fail-Stop-Model*) as well as intentional sabotage (*Byzantine Failures*). However, a much more prevalent "failure" has largely been ignored: Selfishness. Though the topic receives due attention lately (for instance [1],[3]), game-theoretic analyses of the performance of distributed algorithms remain the exception rather than the norm.

Shirking is a specific manifestation of selfishness where tasks are not performed simply because the performer does not benefit from performing them, regardless of their potential benefit for the whole. To the best of our knowledge, there has been no formal model, let alone a game theoretic analysis, of that phenomenon – developping a formal model for shirking and analyzing its properties has therefore been the objective of this semester project. (While analyzing shirking itself is trivial, things begin to become interesting once commonly employed counter-measures are taken into account, players can not reject tasks individually and have limited capabilities – so fear not, there is some meat in this :))

We begin by giving an introduction to shirking and commonly employed counteractives, then proceed to develop an informal and then a formal model. In Section 2, we prove the game's convergence behaviour; in Section 3, we characterize the social cost and prove a bound for the game's Price of Anarchy. In Section 4, we conclude. There is also an appendix listing standard game-theoretic definitions used in this paper.

1.1 Background

Frequently, a group of people is confronted with a set of tasks whose completion benefits everyone, but every member can decide independently whether to take action or not, taking into account only the benefits and costs for himself. As a consequence, tasks whose execution would benefit the group may not be performed. Typically, the group is aware of that and tries to mitigate that situation by penalizing shirking. This is usually done by using one or both of the following approaches:

1.1.1 Punishment

If a player shirks, he is punished. The punishment's cost is payed by the shirker. For instance, if someone refuses to be drafted into the military, he may be put in prison. This incurs a cost that is essentially payed by the offender.

1.1.2 Distribution of Damage

Society defines who pays for the damage incurred by unhandled tasks. If society can find out who could have averted the damage, it can hold these responsible, i.e. require them to pay for the damage. We call this *informed oversight.*

On the other hand, if it is too expensive or simply infeasible to identify the responsible players, every player contributes his share to the damage. We call this *oblivios oversight.*

1.2 Informal Model

Every player is tasked with some responsibilities with associated costs, but can opt out (shirk). If shirking is punished, this incurs a fixed cost. The responsibility for a task can be shared by several players, in which case the cost of the task is divided evenly between those who do not shirk[1]. If all responsible players shirk, the task incurs a *damage* which is distributed evenly among all responsible players in informed oversight or all players in oblivious oversight.

1.3 Formal Model

A game is characterized by the set of players P, the set of tasks T, the can-do relation $\sim\subseteq P \times T$, the shirk cost $\lambda \geq 0$, the task cost $c(t) \geq 0$, and the task damage $d(t) \geq 0$.

1.3.1 Strategy

A strategy is characterized by the function $P \rightarrow \{0,1\}$, where $s(p) = 1$ iff p shirks. Note that this implies that we consider deterministic strategies only.

In this section, s, s' denote any strategy. To shorten notation, we define:

$$
\begin{array}{rcl}
s \leq s' & \Leftrightarrow & \forall p \in P \cdot s(p) \leq s'(p) \\
s < s' & \Leftrightarrow & s \leq s' \wedge s \neq s' \\
S(s) & = & \{p \in P | s(p) = 1\} \\
R(s) & = & P \setminus S(s) \\
R(s)(t) & = & \{p \in R(s) | p \sim t\} \\
U(s) & = & \{t \in T | R(s)(t) = \emptyset\}
\end{array}
$$

Also, square brackets following a function denote a function update:

$$
s[p' \rightarrow b](p) = \begin{cases} b & \text{if } p = p' \\ s(p) & \text{otherwise} \end{cases}
$$

[1]This also models tasks that must be performed by one player if the poor guy is selected at random among the eligible players.

and square brackets not following a function denote indicator functions:

$$[b = \pi] = \begin{cases} 1 & \text{if } b = \pi \\ 0 & \text{otherwise} \end{cases}$$

1.3.2 Cost

If p is a player and t is a task, p's contribution to t (for oblivious oversight) is given by

$$con(s)(p,t) = \begin{cases} \frac{d(t)}{|P|} & \text{if } R(s)(t) = \emptyset \\ \frac{c(t)}{|R(s)(t)|} & \text{if } R(s)(t) \neq \emptyset \wedge p \in R(s)(t) \\ 0 & \text{otherwise} \end{cases}$$

and (for informed oversight) by

$$con(s)(p,t) = \begin{cases} \frac{d(t)}{|P|} & \text{if } \{p|p \sim t\} = \emptyset \\ \frac{d(t)}{|\{p|p \sim t\}|} & \text{if } p \sim t \wedge R(s)(t) = \emptyset \\ \frac{c(t)}{|R(s)(t)|} & \text{if } p \in R(s)(t) \\ 0 & \text{otherwise} \end{cases}$$

The cost for p is given by

$$cost(s)(p) = s(p)\lambda + \sum_{t \in T} con(s)(p,t)$$

and his incentive to shirk by

$$\begin{aligned} inc(s)(p) \quad &:= \quad cost(s[p \to 0], p) \\ &- \quad cost(s[p \to 1], p) \end{aligned}$$

1.3.3 Social Cost

As usual, we define:

$$\begin{aligned} soccost(s) \quad &= \quad \sum_{p \in P} cost(s)(p) \\ socopt \quad &= \quad \min_s soccost(s) \end{aligned}$$

For our game, this means:

$$\begin{aligned} soccost(s) \quad &= \quad \sum_{t \in U(s)} d(t) + \sum_{t \in T \setminus U(s)} c(t) + |S(s)|\lambda \\ &= \quad \sum_{t \in U(s)} (d(t) - c(t)) + \sum_{t \in T} c(t) + |S(s)|\lambda \end{aligned}$$

1.4 Equivalence of Flavors

Now that we have defined the various flavors, we take care that flavors are not redundant.

Lemma 1 *Informed oversight with punishment is not more expressive than informed oversight without punishment, i.e. for every game in informed oversight with punishment there is an equivalent game in informed oversight without punishment.*

Proof Sketch: Let $G = (P, T, \sim, \lambda, c', d')$ be any game in informed oversight with punishment. Without loss of generality, we assume $P \cap T = \emptyset$. We take the smallest extension of that game satisfying

$$T \subseteq P$$

$$p \sim p$$
$$d'(p) = \lambda$$
$$c'(p) = 0$$

set $\lambda = 0$ and call it G'. Then,

$$cost_G(s)(p) = cost_{G'}(s)(p)$$

i.e.

$$G \equiv G'$$

2 The Nash-Iteration

In general, it is not trivial that a game has pure Nash Equilibria, and even if it does, Nash-Iteration may not stabilize for some starting strategies.

2.1 Does It Converge?

Lemma 2 (Convergence) *For any flavor of the model, every Nash-Iteration converges.*

Proof Outline: We derive the contribution of any task to the player's preferred choices, transform this into a potential function, which we prove to be strictly monotonically increasing during the iteration.

Proof Idea: If a player's change in strategy affected only himself, we could use the following potential function:

$$\Phi(s) = \sum_{p|s(p)=1} inc(s)(p)$$

Alas, a player's change also affects the incentive of other players (and consequently the potential) in a non-trivial, sometimes negative manner. We handle this by extending the potential function to include a term that cancels the change's side effects on the potential.

Proof: The proofs in the various models are very similar; we present only the proof for informed oversight without shirk punishment. We describe the adjustments needed for oblivious oversight or shirk punishments at the end of the proof.

A player's preferred choice is governed by the difference in cost between his two choices. We define his incentive to shirk as that difference:

$$\begin{aligned} inc(s)(p) &:= cost(s[p \to 0], p) \\ &- cost(s[p \to 1], p) \end{aligned}$$

We observe that $s(p)$ does not influence the incentive. To keep notation simple, we define:

$$\begin{aligned} s_0 &= s[p \to 0] \\ s_1 &= s[p \to 1] \end{aligned}$$

Additionally substituting the definition of cost, we get

$$\begin{aligned} inc(s)(p) &:= \sum_t con(s_0)(p,t) \\ &- \sum_t con(s_1)(p,t) \\ &= \sum_t \underbrace{con(s_0)(p,t) - con(s_1)(p,t)}_{=:ico(s,p,t)} \end{aligned}$$

ico stands for incentive contribution. Obviously, if $p \not\sim t$, $ico(s,p,t) = 0$. For the case $p \sim t$, let

$$C(t) = \{p | p \sim t\}$$

We observe

$$con(s_0)(p) = \frac{c(t)}{|R(s_0)(t)|}$$

$$con(s_1)(p) = \begin{cases} \frac{d}{|C(t)|} & \text{if } R(s_1)(t) = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Beeing equipped with all neccessary ingredients, we define our potential function:

$$\Phi(s) = \sum_t [R(s)(t) = \emptyset] - (|C(t)| - 1)\frac{d(t)}{|C(t)|} + \sum_{r=1}^{|R(s)(t)|} (|C(t)| - r)\frac{c(t)}{r(r+1)} + \sum_{p|s(p)} ico(s,p,t)$$

It is worth noting that the only strategy dependent subexpressions are $|R(s)(t)|$, $ico(s, p, t)$ and $s(p)$.

It remains to show that

$$s \xrightarrow{N} s' \Rightarrow \Phi(s) < \Phi(s')$$

⌜**let** s, s' **such that** $s \xrightarrow{N} s'$

By the definition of $\xrightarrow{N}$, there is a p_0 such that

$$s[p_0 \rightarrow s'(p_0)] = s'$$

Let p_0 be this player. Obviously, $s \neq s'$ and therefore

$$s(p_0) \neq s'(p_0)$$

Let

$$\begin{aligned} s_0 &= \min(s, s') \\ s_1 &= \max(s, s') \end{aligned}$$

We then have

$$\Phi(s) = [s(p_0) = 1] inc(s)(p_0) + \sum_t tico_t(s)$$

where

$$\begin{aligned} tico_t(s) &= [R(s)(t) = \emptyset](|C(t)| - 1)\frac{d(t)}{|C(t)|} \\ &+ \sum_{r=1}^{|R(s)(t)|+1} (|C(t)| - r - 1)\frac{c(t)}{r(r+1)} \\ &+ \sum_{\substack{p|s(p) \\ p \neq p_0}} ico(s, p, t) \end{aligned}$$

Let t be any task and $tico = tico_t$. We are now going to prove

$$tico(s) = tico(s')$$

⌜**case** $p_0 \sim t$

$$\sum_{\substack{p|s(p)\\p\neq p_0}} ico(s,p,t) = \sum_{\substack{p|s(p)\\p\sim t\\p\neq p_0}} ico(s,p,t) + \sum_{\substack{p|s(p)\\p\not\sim t\\p\neq p_0}} \underbrace{ico(s,p,t)}_{0}$$

$$ico(s,p,t) = ico(s_0,p,t)$$

⌜**case** $R(s_1)(t) = \emptyset$

$$R(s_0)(t) = \{p_0\}$$

$$\begin{aligned}
tico(s_0) &= \sum_{r=1}^{|R(s_0)(t)|} (|C(t)|-r)\frac{c(t)}{r(r+1)} + \sum_{\substack{p|s_0(p)\\p\sim t\\p\neq p_0}} ico(s_0,p,t) \\
&= (|C(t)|-1)\frac{c(t)}{2} + \sum_{\substack{p|s_0(p)\\p\sim t\\p\neq p_0}} \left(\frac{c(t)}{2}\right) \\
&= (|C(t)|-1)\frac{c(t)}{2} + (|C(t)|-1)\frac{c(t)}{2} \\
&= (|C(t)|-1)c(t) \\
&= (|C(t)|-1)\frac{d(t)}{|C(t)|} + (|C(t)|-1)\left(c(t) - \frac{d(t)}{|C(t)|}\right) \\
&= (|C(t)|-1)\frac{d(t)}{|C(t)|} + \sum_{\substack{p|s_1(p)\\p\sim t\\p\neq p_0}} \left(c(t) - \frac{d(t)}{|C(t)|}\right) \\
&= (|C(t)|-1)\frac{d(t)}{|C(t)|} + \sum_{p|s_1(p)} ico(s_1,p,t) \\
&= tico(s_1)
\end{aligned}$$

⌞

⌜**case** $R(s_1)(p,t) \neq \emptyset$

Let $w = |R(s_0)(p,t)|$. We have

$$\begin{aligned}
&tico(s_0) \\
= &\sum_{r=1}^{|R(s_0)(t)|} (|C(t)|-r)\frac{c(t)}{r(r+1)} + \sum_{p|s_0(p)} ico(s_0,p,t)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{r=1}^{|R(s_1)(t)|} (|C(t)| - r)\frac{c(t)}{r(r+1)} \; + \; (|C(t)| - w)\frac{c(t)}{w(w+1)} + \sum_{\substack{p|s_0(p)\\ p\sim t\\ p\neq p_0}} \left(\frac{c(t)}{w+1}\right) \\
&= \sum_{r=1}^{|R(s_1)(t)|} (|C(t)| - r)\frac{c(t)}{r(r+1)} \; + \; (|C(t)| - w)\frac{c(t)}{w(w+1)} + (|C(t)| - w)\left(\frac{c(t)}{w+1}\right) \\
&= \sum_{r=1}^{|R(s_1)(t)|} (|C(t)| - r)\frac{c(t)}{r(r+1)} \; + \; (|C(t)| - w)\left(\frac{c(t)}{w}\right) \\
&= \sum_{r=1}^{|R(s_1)(t)|} (|C(t)| - r)\frac{c(t)}{r(r+1)} \; + \sum_{\substack{p|s_1(p)\\ p\sim t\\ p\neq p_0}} ico(s_1, p, t) \\
&= tico(s_1)
\end{aligned}
$$

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⌜**case** $p_0 \not\sim t$

$$R(s,t) = R(s',t)$$
$$ico(s,p,t) = ico(s',p,t)$$
$$tico(s) = tico(s')$$

⌞

We have proven
$$tico(s) = tico(s')$$

⌜**case** $s(p_0) = 0$

$$s'(p_0) = 1$$

Since $s \xrightarrow{N} s'$,
$$inc(s)(p_0) > 0$$

Also,

$$
\begin{array}{lcl}
\Phi(s) + inc(s)(p_0) & = & \sum_t tico_t(s) \; + \; inc(s)(p_0) \\
 & & \qquad\qquad\quad \| \\
\Phi(s') & = & \sum_t tico_t(s') \; + \; inc(s')(p_0)
\end{array}
$$

Therefore
$$\Phi(s) < \Phi(s')$$

⌞

⌜**case** $s(p_0) = 1$

$$s'(p_0) = 0$$

Since $s \xrightarrow{N} s'$,

$$inc(s)(p_0) < 0$$

Also,

$$\begin{array}{lcl} \Phi(s) & = & \sum_t tico_t(s) \ + \ inc(s)(p_0) \\ & & \qquad\qquad\qquad \| \\ \Phi(s') + inc(s')(p_0) & = & \sum_t tico_t(s') \ + \ inc(s')(p_0) \end{array}$$

Therefore

$$\Phi(s) < \Phi(s')$$

⌞

⌞

The proof can be adapted to oblivious oversight by replacing the denominator below $c(t)$ to $|P|$ in the whole proof. It can be adapted to shirk punishment by adding $\sum_p s(p)\lambda$ to $\Phi(s)$: The invariance of *tico* is not affected, and since the only $s(p)$ changing in the iteration is $s(p_0)$, which is contained in $inc(s)(p,t)$, we still have $\Phi(s') = \Phi(s) + |inc(s)(p_0)|$. q.e.d.

It is worth noting that the changing player's absolute incentive is equal to the increase in the potential function. Therefore, the bound on the number iteration steps imposed by this theorem is lower if players with better reasons to change change first. Perhaps this holds for the number of steps itself. This would be interesting to investigate further, but is outside the scope of this project.

Corollary 3 *For every game, there exists a pure Nash Equilibirium, i.e. a deterministic strategy such that every player is content with her choice.*

This is quite surprising.

2.2 Speed of Convergence

While studying the speed of convergence was not our primary objective, the following theorem gives some insight into a restriced subclass of Nash-Iterations.

Lemma 4 *If* $\forall t \in T \cdot d(t) \leq \frac{|C(t)|}{2} c(t)$ *and* $b \in \{0,1\}$ *and* s_0 *such that* $\forall p \in P \cdot s_0(p) = b$*, Nash-Iteration starting in* s_0 *reaches a fixpoint in at most* $|P|$ *steps. (informed oversight, shirk punishment)*

The same, except that the bound on $d(t)$ *is* $\forall t \in T \cdot d(t) \leq \frac{|P|}{2} c(t)$*, holds for oblivious oversight.*

Proof: The two proofs are very similar, we give only the first. The second one could be obtained by replacing $|P|$ with $|C(t)|$ wherever it occurs. For the same reason, we elaborate only on the case $b = 0$; the proof for $b = 1$ is analogous.

We define a potential function on strategies:

$$\Phi(s) := \sum_{p \in P} s(p)$$

Let p be any player. We prove that $s_p = 1 \Rightarrow inc(s,p) > 0$ is an iteration invariant and $\Phi(s_i) = i$ using induction along the iteration.

Initially,

$$s(p) = 0$$

$$\Phi(s_0) = 0$$

$$s(p) = 1 \Rightarrow inc(s,p) > 0$$

For the induction step, let $s = s_i$ be the current strategy and $s' = s_{i+1}$ its successor. Since it is a Nash-Iteration, s differs from s' for exactly one player, which we denote by up. Let $x = s(up)$ and $x' = s'(up)$

We therefore have:

$$x \neq x'$$

$$s' = s[up \to x']$$

$$cost(s', up) < cost(s, up)$$

Assume $x = 1$. Then,

$$x' = 0$$

By the induction hypothesis, we get

$$s(up) = 1 \Rightarrow 0 < inc(s, up)$$

$$0 < inc(s, up)$$

By definition of inc:

$$inc(s, up) = cost(s[up \to 0], up) - cost(s[up \to 1], up)$$

impying

$$cost(s[up \to 1], up) < cost(s[up \to 0], up)$$

$$cost(s, up) < cost(s', up)$$

but

$$cost(s', up) < cost(s, up)$$

therefore $x' = 1$ and

$$\Phi(s') = \Phi(s) + 1 = i + 1$$

It remains to justify the induction hypothesis:

assume $s'(p) = 1$

case $p = up$

$$cost(s', up) < cost(s, up)$$

$$cost(s'[up \to 1], up) < cost(s'[up \to 0], up)$$

$$inc(s', up) = cost(s'[up \to 0], up) - cost(s'[up \to 1], up) > 0$$

$$inc(s', p) > 0$$

case $p \neq up$

$$s(p) = s'(p) = 1$$

By the induction hypothesis:

$$s(p) = 1 \Rightarrow inc(s)(p) > 0$$

$$inc(s)(p) > 0$$

By the definitions of *inc* and *cost*:

$$\begin{array}{rcrcl}
inc(s)(p) & = & cost(s[p \to 0], p) & - & cost(s[p \to 1], p) \\
 & & & \| & \\
 & & cost(s[p \to 0], p) & - & cost(s, p) \\
 & & & \| & \\
 & & -\lambda + \sum_{t \in T} con(s[p \to 0])(p, t) & - & con(s)(p, t) \\
 & & & ? & \\
 & & -\lambda + \sum_{t \in T} con(s'[p \to 0])(p, t) & - & con(s')(p, t) \\
 & & & \| & \\
 & & cost(s'[p \to 0], p) & - & cost(s', p) \\
 & & & \| & \\
inc(s')(p) & = & cost(s'[p \to 0], p) & - & cost(s'[p \to 1], p)
\end{array}$$

To supply the missing link ?, we now prove the lemma

$$\begin{array}{rl}
 & con(s[p \to 0])(p, t) - con(s)(p, t) \\
\leq & con(s'[p \to 0])(p, t) - con(s')(p, t)
\end{array}$$

by case distinction. Let t be any task.

case $p \not\sim t$

$$\begin{array}{rl} & con(s[p \to 0])(p,t) - con(s)(p,t) \\ = & 0 \\ = & con(s'[p \to 0])(p,t) - con(s')(p,t) \end{array}$$

⌞

⌜**case** $p \sim t$

⌜**case** $R(s)(t) = \emptyset$

$$R(s')(t) = \emptyset$$

$$\begin{array}{rrcl} & con(s[p \to 0])(p,t) & - & con(s)(p,t) \\ = & c(t) & - & \frac{d(t)}{|C(t)|} \\ = & con(s'[p \to 0])(p,t) & - & con(s')(p,t) \end{array}$$

⌞

⌜**case** $R(s)(t) = \{up\}$

$$R(s')(t) = \emptyset$$

$$\begin{array}{rrcl} & con(s[p \to 0])(p,t) & - & con(s)(p,t) \\ = & \frac{c(t)}{2} & - & 0 \\ \overset{!}{\leq} & c(t) & - & \frac{d(t)}{|C(t)|} \\ = & con(s'[p \to 0])(p,t) & - & con(s')(p,t) \end{array}$$

⌞

⌜**case** $\emptyset \neq R(s)(t) \neq \{up\}$

$$R(s')(t) \subseteq R(s)(t)$$

$$\begin{array}{rrcl} & con(s[p \to 0])(p,t) & - & con(s)(p,t) \\ = & \frac{c(t)}{|R(s)(t) \cup \{p\}|} & - & 0 \\ \leq & \frac{c(t)}{|R(s')(t) \cup \{p\}|} & - & 0 \\ = & con(s'[p \to 0])(p,t) & - & con(s')(p,t) \end{array}$$

⌞

⌞

Having proved the lemma, we know

$$0 < inc(s)(p) \leq inc(s')(p)$$

$$inc(s')(p) > 0$$

∟

$$inc(s')(p) > 0$$

∟

$$s'(p) = 1 \Rightarrow inc(s')(p) > 0$$

i.e. the induction hypothesis holds for s'.

We have proven:

$$\Phi(s_i) = i$$

However,

$$\Phi(s_i) \leq |P|$$

therefore

$$i \leq |P|$$

q.e.d.

Note that this bound does not hold in general. For instance, consider the game without shirk punishment given by

t	$c(t)$	$d(t)$	$C(t)$
1	2	8	$\{1,2\}$
2	4	0	$\{2\}$

and the sequence of strategies

i	$s_i(1)$	$s_i(2)$
0	0	0
1	1	0
2	1	1
3	0	1

This sequence is a Nash-Iteration starting in a homogenous strategy, but takes longer than $|P|$ steps to reach a fixpoint.

3 The Social Cost

Definition 1 (Reasonable Game) *A game is reasonable iff every task makes sense, i.e.* $\forall t \in T \cdot d(t) \geq c(t)$.

Lemma 5 *If the game (in informed or oblivious oversight) is reasonable,* $soccost(s)$ *is monotonic with respect to* s.

Proof Sketch: All summands are monotonic in s, therefore, so is the sum.

Corollary 6 *If the game is reasonable and* $\sim$ *is surjective,*

$$socopt = \sum_{t \in T} c(t)$$

3.1 For Complete Graphs

We assume shirk-cost of 1 in this section. If $\sim = P \times T$, we can assume $|T| = 1$ without loss of generality. Also, all players beeing equal except for their strategy, we can characterize strategies by the number of shirkers s. Also, we define $n = |P|$ and $t = n - s$.

Let's characterize the possible equilibria:

assume s **is a Nash-Equilibrium**

case $0 < s$

$$\exists p \in P \cdot s(p) = 1$$

case $0 < s < n$

$$\frac{c}{t+1} \geq 1$$

case $s = n$

$$c \geq 1 + \frac{d}{n}$$

case $s < n$

$$\exists p \in P \cdot s(p) = 0$$

case $s < n - 1$

$$\frac{c}{t} \leq 1$$

case $s = n - 1$

$$c \leq 1 + \frac{d}{n}$$

It is easy to see that these conditions are also sufficient for an equilibrium.

Therefore, for $n \geq 2$, the conditions for s to be an equilibrium are

s	condition
0	$c \leq n$
$n-1$	$2 \leq c \leq 1 + \frac{d}{n}$
n	$1 + \frac{d}{n} \leq c$

The social cost is given by

$$soccost(s) = \begin{cases} c + s & \text{if } s < n \\ d + n & \text{otherwise} \end{cases}$$

and the social optimum by

$$socopt = \min(c, d + n)$$

Analyzing the cost, we get (for $n \geq 2$):

		equil.	*equil.cost*	*socopt*	*POA*	*POS*
$c < 1$		$\{0\}$	$\{c\}$	c	1	1
$1 \leq c < 2$	$d \leq n(c-1)$	$\{0, n\}$	$\{c, d+n\}$	c	$\frac{n}{c} \leq \frac{d+n}{c} \leq n$	1
$1 \leq c < 2$	$n(c-1) < d$	$\{0\}$	$\{c\}$	c	1	1
$2 \leq c \leq n$	$d < n(c-1)$	$\{0, n\}$	$\{c, d+n\}$	c	$\frac{n}{c} \leq \frac{d+n}{c} \leq n$	1
$2 \leq c \leq n$	$n(c-1) < d$	$\{0, n-1\}$	$\{c, c+n-1\}$	c	$\frac{c+n-1}{c}$	1
$n < c$	$d \leq c - n$	$\{n\}$	$\{d+n\}$	$d+n$	1	1
$n < c$	$c - n \leq d < n(c-1)$	$\{n\}$	$\{d+n\}$	c	$1 \leq \frac{d+n}{c} \leq n$	$1 \leq \frac{d+n}{c} \leq n$
$n < c$	$n(c-1) < d$	$\{n-1\}$	$\{c+n-1\}$	c	$\frac{c+n-1}{c}$	$\frac{c+n-1}{c}$

where *equil* is the set of Nash Equilibria, *equil.cost* is the set of their costs, *POA* the price of anarchy and *POS* the price of stability (the bounds are the suprema/infima over the values of d).

We observe that even this simple and highly symmetric (and thus fair) scenario features really bad instances with a *POS* of $|P|$.

3.2 The Price of Anarchy

For simplicity's sake, we restrict ourselves to informed oversight without punishment and reasonable games in this section. Since computing the Price of Anarchy appears to be hard, we turn our attention to upper bounds to POA for classes of games.

Lemma 7 (Structure of Most Expensive Instance Equilibrium) *Among all reasonable games for some* $(P, T, \sim, \lambda = 0, c')$*, an equilibirum with maximum social cost satisfies:*

$$R(s) = \emptyset$$

Proof: Let C be such a class of games. Since there is an equilibrium for every game, i.e. there are equilibria for games in C, it sufficices to prove that for every equilibrium s for a game $G \in C$, there is a game $G' \in C$ and a totally shirking equilibrium s' for G' such that

$$soc.cost(s) \leq soc.cost'(s')$$

Let s be any equilibrium strategy of a reasonable game d and H the set of handled tasks given s, i.e.

$$H = \{t \in T | \exists p \in P \cdot s(p) = 0 \wedge p \sim t\}$$

Then, consider the game d' defined by

$$d'(t) = \begin{cases} c(t) & \text{if } t \in H \\ d(t) & \text{otherwise} \end{cases}$$

Obviously, it is reasonable and thus in C. Also, it is easy to see from the definition of *con* that

$$con'(s) = con(s)$$

and therefore

$$soc.cost'(s) = soc.cost(s)$$

Let s' be the totally shirking strategy. Since $s \leq s'$,

$$soc.cost'(s) \leq soc.cost'(s')$$

Therefore,

$$soc.cost(s) \leq soc.cost'(s')$$

We will now prove that $s' \in equil(d')$. Let p be any player.

case $s(p) = 1$

Since $s \in equil(d)$,

$$0 \leq inc(s)(p)$$

Since damage discourages shirking (i.e. *inc* is monotonic decreasing in d)

$$inc(s)(p) \leq inc'(s)(p)$$

Moreover, no additionally shirking player p_0 discourages shirking, as any task t attached to p_0 is in H and therefore $d(t) = c(t)$:

$$inc'(s)(p) \leq inc'(s')(p)$$

Putting it all together:

$$0 \leq inc'(s')(p)$$

case $s(p) = 0$

Let t be any task.

⌜**case** $p \sim t$

$$t \in H$$

$$d(t) = c(t)$$

$$con(s'[p \to 1])(p,t) \leq \frac{d(t)}{|C(t)|} \leq \frac{c(t)}{|R(s'[p \to 0])(t)|} \leq con(s'[p \to 0])(p,t)$$

$$0 \leq ico(s')(p,t)$$

⌞

⌜**case** $p \not\sim t$

$$con(s'[p \to 1])(p,t) = con(s'[p \to 0])(p,t)$$

$$0 = ico(s')(p,t)$$

⌞

$$inc(s')(p) = \sum_t ico(s', p, t) \geq 0$$

⌞

Therefore, $\forall p$,

$$inc'(s')(p) \geq 0$$

and consequently

$$s' \in equil(d')$$

q.e.d.

Lemma 8 (POA For Classes of Reasonable Games) *If $\sim$ is surjective and C is the class of reasonable games sharing $(P, T, \sim, 0, c)$, the maximum POA over the games in C is the average task degree, weighted by the task cost:*

$$\max_{G \in C} POA(G) = \frac{\sum_{t \in T} c(t) * |\{p | p \sim t\}|}{\sum_{t \in T} c(t)}$$

Proof: The previous lemma implies that we can assume the most expensive equilibrium strategy to be totally shirking without loss of generality. Let s be a totally shirking strategy. s is an equilibrium iff $\forall p \in P$

$$cost(s)(p) \leq cost(s[p \to 0])(p)$$

i.e.

$$\sum_{t|p \sim t} \frac{d(t)}{|C(t)|} \leq \sum_{t|p \sim t} c(t)$$

⌜**if** s **is an equilibrium,**

$$\forall p \in P \cdot cost(s)(p) \leq \sum_{t|p\sim t} c(t)$$

$$\begin{aligned}
\sum_{p\in P} cost(s)(p) &\leq \sum_{p\in P}\sum_{t|p\sim t} c(t) \\
soc.cost(s) &\leq \sum_{t\in T}\sum_{p\sim t} c(t) \\
&\leq \sum_{t\in T} c(t) * |\{p|p \sim t\}|
\end{aligned}$$

└

Since $|C(t)| \geq 1$, the game defined by $d(t) = c(t) * |C(t)|$ is in C. It satisfies the equilibrium condition and thus the above inequality tightly. We therefore have

$$\max_{s,G\in C} soc.cost_G(s) = \sum_{t\in T} c(t) * |\{p|p \sim t\}|$$

By Corollary 6, we know

$$\forall G \in C \cdot soc.opt_G = \sum_{t\in T} c(t)$$

Therefore

$$\max_{G\in C} POA(G) = \max_{\substack{s\in equil(G)\\ G\in C}} \frac{soc.cost_G(s)}{soc.opt_G} = \frac{\sum_{t\in T} c(t) * |\{p|p \sim t\}|}{\sum_{t\in T} c(t)}$$

q.e.d.

Corollary 9 *For any reasonable game with task-degree k,*

$$POA(G) \leq k$$

3.3 The Price of Stability

Corollary 10 *There is a game where the Price of Stability is equal to the number of players.*

Proof: See our treatment of the complete bipartite graph. For $n < c$ and $d = n(c-1) - \varepsilon$, the social cost of the only equilibrium is cn, but the social optimum is n.

4 Conclusion

We have presented a formal model for the process of shirking under various counteractive measures, which we modeled using different model flavours. We have proven that the game's Nash-Iteration always converges (regardless of what counteractive measures are used), and that, if some conditions are met, it does so in time linear in the number of players. We have comprehensively described the behaviour for games featuring a maximum can-do relation, showing that there are instances for which both the Price of Anarchy and the Price of Stability equal the number of players. We have also proven that the can-do relation's weighted average task-degree is a tight bound for the Price of Anarchy for the subclass of reasonable games.

4.1 Further Work

The game beeing freshly formalized, there are numerous open problems. Some of the more intersting ones include:

- a characterization for POA/POS for arbitrary reasonable games
- a bound on convergence speed for large damages
- bounds on social cost/POA/POS for non-reasonable games
- the influence of shirk punishment size on social cost

A Standard Definitions

Definition 2 (Nash-Iteration) *Let* $\stackrel{N}{\rightarrow}$ *be the smallest mapping between strategies such that*

$$cost(s)(p) > cost(s[p \rightarrow b])(p) \Rightarrow s \stackrel{N}{\rightarrow} s[p \rightarrow b]$$

If $s \stackrel{N}{\rightarrow} s'$, s' *is called Nash-Successor of* s[2].

Nash-Iteration is the process obtained by iteratively going to a Nash-Successor in every timestep or stay at the same place if the current strategy has no Nash-Successor.

Definition 3 (Nash-Equilibrium) *A strategy* s *is called Nash-Equilibrium iff it is a fixpoint in the Nash-Iteration, i.e.*

$$\nexists s' \cdot s \stackrel{N}{\rightarrow} s'$$

Definition 4 (Convergence of Nash-Iteration) *We say that Nash-Iteration converges for a game iff every Nash-Iteration converges for that game.*

Lemma 11 *Nash-Iteration converges iff* $\stackrel{N}{\rightarrow}$ *is acyclic.*

Proof: Trivial.

Introduced by Koutsopias and Papadimitriou [2], the price of anarchy measures the worst case cost impact of anarchy:

Definition 5 (Price of Anarchy) *The Price Of Anarchy is defined as the ratio of social cost of the most expensive equilibrium strategy to the least expensive strategy:*

$$POA = \max_{s \in equil} \frac{soc.cost(s)}{soc.opt}$$

Definition 6 (Price Of Stability) *Complementary to the Price Of Anarchy, the Price Of Stability is defined as the ratio of social cost of the* least *expensive equilibrium strategy to the least expensive strategy:*

$$POS = \min_{s \in equil} \frac{soc.cost(s)}{soc.opt}$$

[2]Note that a state may have 0 or more than 1 Nash-Successor.

B Related Work

1. J. Aspnes, K. Chang and A. Yampolskiy. Inoculation strategies for victims of viruses and the sum-of-squares partition problem. *Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 43–52, 2005.

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3. H. Kunreuther and G. Heal. Interdependent security. *Journal of Risk and Uncertainty (Special Issue on Terrorist Risks)*, 2003.