An Optimal Bound for the MST Algorithm to Compute Energy Efficient Broadcast Trees in Wireless Networks

Christoph Ambühl*

Department of Computer Science,
The University of Liverpool
christoph@csc.liv.ac.uk

Abstract. Computing energy efficient broadcast trees is one of the most prominent operations in wireless networks. For stations embedded in the Euclidean plane, the best analytic result known to date is a 6.33-approximation algorithm based on computing an Euclidean minimum spanning tree. We improve the analysis of this algorithm and show that its approximation ratio is 6, which matches a previously known lower bound for this algorithm.

1 Introduction

Multi-hop wireless networks [14] require neither fixed, wired infrastructure nor predetermined interconnectivity. In particular, ad hoc networks [11, 17] are the most popular type of multi-hop wireless networks. An ad hoc wireless network is built of a bunch of radio stations. The links between them are established in a wireless fashion using the radio transmitters and receivers of the stations.

In order to send a message from a station s to a station t, station s needs to emit the message with enough power such that t can receive it. In the model, the power P_s required by a station s to transmit data to station t must satisfy the inequality

$$\frac{P_s}{\mathrm{dist}(s,t)^\alpha} \;>\; \gamma. \tag{1}$$

The term $\mathrm{dist}(s,t)$ denotes the distance between s and t, and $\alpha \geq 1$ is the **distance-power gradient**, and $\gamma \geq 1$ is the **transmission-quality** parameter. In an ideal environment (i.e. in the empty space) it holds that $\alpha = 2$ but it may vary from 1 to more than 6 depending on the environment conditions of the location of the network (see [19]).

In ad hoc networks, a power value is assigned to each station. These values, according to Equation (1), determine the so-called **range** of each station. The range of a station s is the area in which stations can receive all messages sent by s.

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Using the ranges, one can determine the so-called **transmission graph** $G = (S, A)$. The vertex set S is the set of stations, and the directed edge from s to t is in A if and only if t is within the range of s.

All stations in the range of a station i can receive messages sent by i. The minimal range needed for station i to establish all its out-going connections in G is therefore

$$r_G(i) := \max_{j \in \Gamma_G(i)} \mathrm{dist}(i, j). \tag{2}$$

where $\Gamma_G(i)$ denotes the set of out-neighbors of station i in G. The total power needed to establish all connections in G is therefore

$$\mathrm{power}(G) := \sum_{i \in V} \gamma \cdot r_G(i)^\alpha, \tag{3}$$

Since the value of γ does not influence the relative quality of the solutions, we assume $\gamma = 1$ for the rest of the paper. In this paper we address the following problem:

*Problem 1 (Energy Efficient Broadcast Tree Problem (**EEBT**)).* Let S be a set of stations represented by points from the Euclidean plane. That is, the distance function becomes $\mathrm{dist}(s, t) := |st|$, where $|st|$ is the Eulidean distance between s and t. One of the stations is called the source station s. The goal is to find the transmission graph G which minimizes $\mathrm{power}(G)$ and contains a directed spanning tree rooted at s (a branching from s).

The relevance of this problem is due to the fact that any transmission graph satisfying the above property allows the source station to perform a **broadcast** operation. Broadcast is a task initiated by the source station to transmit a message to all stations in the wireless network: This task constitutes a major part of real life multi-hop radio networks [1, 2, 7].

The **EEBT** Problem is known to be $\mathcal{NP}$-hard [4, 3]. Furthermore, if the dist function is arbitrary, the problem cannot be approximated with a logarithmic factor unless $\mathcal{P} = \mathcal{NP}$ [10]. The currently best approximation algorithm for the **EEBT** Problem is as follows.

Algorithm 1 (MSTALG). *The input of the algorithm is a set of stations S represented by points in the Euclidean plane. One of the stations is designated as the source. The algorithm first computes the Euclidean minimum spanning tree (EMST) of the point set S. Then the EMST is turned into a directed EMST by directing all the edges such that there exists a directed path from the source station to all other stations.*

In [21], Wan, Călinescu, Li, and Frieder claimed that **MSTALG** is a 12-approximation. Unfortunately, there is a small error in their paper. The correct analysis yields an approximation ratio of 12.15, as stated by Klasing, Navarra, Papadopoulos, and Perennes in [13]. Independently, Clementi, Crescenzi, Penna, Rossi, and Vocca showed an approximation ratio of 20 for **MSTALG** [4]. Recently, Flammini,Klasing, Navarra, and Perennes [8] showed that **MSTALG** is a 7.6-approximation algorithm. And even more recently, Navarra proved an approximation ratio of 6.33 [18]. In this paper, we show that $\mathbf{MSTALG}$ is a 6-approximation for all $\alpha \geq 2$. This matches the lower bound given in [4] and [21].

Experimental studies reported in [6, 15] show that for most instances, the approximation ratio of $\mathbf{MSTALG}$ is much better than 6. In [16], exact algorithms for **EEBT** have been studied. The **EEBT** problem falls into the class of so-called **range assignment problems**: Find a transmission range assignment such that the corresponding transmission graph G satisfies a given connectivity property Π, and power(G) is minimized (see for example [12, 7]). In [5], the reader may find an exhaustive survey on previous results related to range assignment problems.

Theorem 1. *Let S be a set of points from the unit disk around the origin, with the additional property that the origin is in S. Let $e_1, e_2, \ldots, e_{|S|-1}$ be the edges of the Euclidean minimum spanning tree of S. Then*

$$\mu(S) := \sum_{i=1}^{|S|-1} |e_i|^2 \leq 6.$$

Theorem 1 is the main theorem of this paper. Together with the next lemma, it proves that $\mathbf{MSTALG}$ is a 6-approximation algorithm for the **EEBT** problem.

The problem of giving upper bounds for $\mu(S)$ has already been looked at independently of the **EEBT** problem. Already in 1968, Gilbert and Pollack [9] gave a upper bound of $8\pi/\sqrt{3}$, based on a technique very similar to the one used by Wan et al in [21]. In 1989, Steele gave a bound of 16 based on space filling curves [20].

Lemma 1. *A bound on $\mu(S)$ automatically implies the same bound on the approximation ratio of $\mathbf{MSTALG}$ for $\alpha \geq 2$.*

Up to a few differences concerning the station at the origin of the unit disk, this lemma has already been proven in [21] to obtain the 12.15-approximation. We therefore skip its proof. For the case $\alpha < 2$, Clementi et al have shown that the MST algorithm does not provide a constant approximation ratio [4].

We now sketch the proof of the $\mu(S) \leq 12.15$ bound given in [21]. It works as follows. The cost of each edge e of the MST is represented by a geometric shape called diamonds, shown in Figure 1 on the bottom left. Diamonds consist of two isosceles

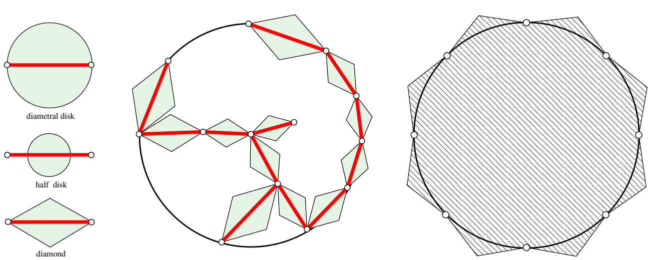

Fig. 1. Proof idea of previous bounds

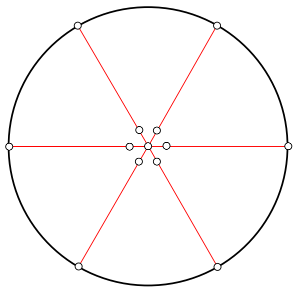

Fig. 2. A worst case example for $\mathbf{MSTALG}$

triangles with an angle of $\frac{2}{3}\pi$. The area of a diamond for an edge e with length $|e|$ is $\lambda \cdot |e|^2$, with $\lambda = \sqrt{3}/6$. Diamonds are considered being open sets. It can be shown that if one puts these diamonds along the edges of an MST as shown in the middle of Figure 1, they do not intersect. It can further be shown that the area of the polygon shown on the right of Figure 1 is an upper bound on the area that can be covered by the diamonds along the MST edges. The area of this polygon is 12.15λ. Therefore one can conclude $\mu(S) \leq 12.15\lambda/\lambda = 12.15$.

Similar bounds can be obtained using diametral disks ($\mu(S) \leq 40$) or half disks ($\mu(S) \leq 20$) [3]. In both cases, one has to give an upper bound on the area generated by these shapes. In the case of diametral disks, this is done using the fact that in any point, at most five diametral disks can intersect. This gives only a very crude bound, which leads to a very crude bound on $\mu(S)$. On the other hand, open half disks do not intersect. But since they are smaller than diamonds, the bound provided by them is worse.

Concerning lower bounds on $\mu(S)$, there is a point set S that attains $\mu(S) = 6$. It is a regular 6-gone with one point in the middle. A lower bound on the approximation ratio of $\mathbf{MSTALG}$ is shown in Figure 2 [21, 4]. The length of the edges of the MST shown in Figure 2 are ε and $1 - \varepsilon$, respectively. We have $\text{opt}(S) = 1$ and $\text{power}(G) = \varepsilon^2 + 6 \cdot (1 - \varepsilon)^2$. Hence for $\varepsilon \to 0$, the ratio between the two becomes 6. This lower bound holds for all values of α. Our analysis will give a matching upper bound for this lower bound for $\alpha \geq 2$. As already stated earlier, $\mathbf{MSTALG}$ does not have a constant approximation ratio for the case $\alpha < 2$ [4].

2 The Main Idea of the Proof of Theorem 1

Among the shapes that do not intersect, diamonds seem to be the best possible geometric shape for this kind of analysis. For a better bound, we need to use larger shapes and we need to deal with the intersection of the shapes more accurately. The shapes used for our new bound are pairs of equilateral triangles. As depicted in Figure 3, the equilateral triangles intersect heavily. We will give a quite accurate bound on the area generated by them.

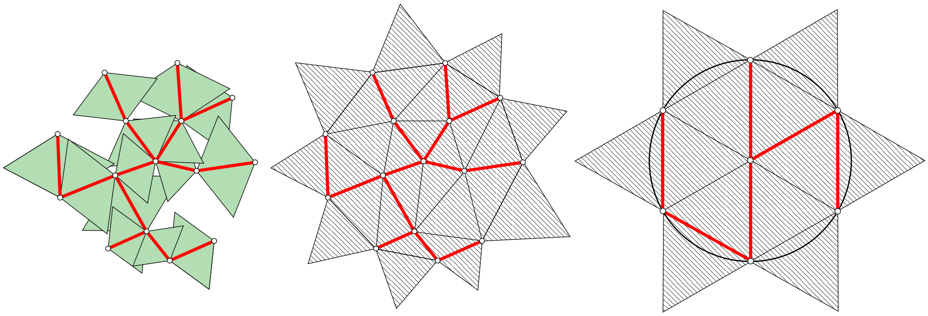

Fig. 3. The total area of the equilateral triangles on the left is bounded by the hatched area in the middle. The point set that maximizes the hatched area is the star shown on the right

A high level description of the proof of our bound is the following. Consider a point set S with n points. Hence, the MST will have $n-1$ edges and therefore, there will be $2(n-1)$ equilateral triangles representing the cost of the MST. Let A_{MST} be the total area generated by these triangles.

In order to obtain an upper bound on A_{MST}, let c be the number of edges of the convex hull of S. By triangulating S, we end up with a planar graph G with n vertices, e edges, and f facets. Let t be the number of triangles of the triangulation. Then the following three equations hold.

$$\left| \begin{array}{rcl} f & = & t+1 \\ 3t+c & = & 2e \\ n+f-e & = & 2 \end{array} \right|$$

The first one simply states that the number of facets is equal to the number of triangles plus the infinite facet. For the second one, we add up the number of edges of all facets. For triangles, this is 3, whereas for the infinite facet, it is c. Since every edge is part of exactly two facets, this sums up to $2e$. The last equation is the Descartes-Euler polyhedral formula [22]. If we solve the system for t, we obtain $t = 2(n-1) - c$. Hence, if we add c equilateral triangles along the convex hull of S as depicted in the center of Figure 3, the number of triangles becomes equal to the number of triangles generated by the MST, as shown on the left side of the figure. Let A_{TRI} be the total area of the triangles within the convex hull of S plus the c additional triangles along the convex hull.

The main idea of the proof is to show that $A_{\mathrm{MST}} \leq A_{\mathrm{TRI}}$. To get an intuitive understanding of it, consider a point set S obtained from the hexagonal grid for which all edges of the triangulation of its convex hull have the same length. In this case, all triangles that are involved in A_{MST} and A_{TRI} are congruent. Furthermore, since their number is equal, it holds $A_{\mathrm{MST}} = A_{\mathrm{TRI}}$. Intuitively, if the edges of the triangulation have different lengths, A_{MST} will be smaller compared to A_{TRI} since the MST will be composed mainly of small edges.

We then conclude the proof by showing that A_{TRI} is maximized by the star configuration depicted on the right of Figure 3. The area of the star is 6λ. Therefore we get $\mu(S) \leq 6\lambda/\lambda = 6$.

3 A Sketch of the Proof of Theorem 1

First, we introduce some notations. The area of an equilateral triangle with side length s will be denoted by $\triangle(s)$. The area of a triangle with edge lengths a, b, and c is denoted by $\triangle(a, b, c)$. Every edge can be partitioned lengthwise into two **half edges**. Both half edges are incident to the same vertices, but each of them is incident to only one facet. Slightly abusing notations, we call the largest side of an obtuse triangle its **hypothenuse**.

Consider the two triangles incident to an edge e. Let α and β be the two angles opposite e in the two triangles. We will call β the **opposite angle of** α.

Consider the MST of S and the Delaunay triangulation of the convex hull of S. Remember that the MST edges are also edges of the Delaunay triangulation. Now choose any edge e of the triangulation. Consider the unique cycle that is formed by adding e to the MST. This cycle and its (finite) interior is called a **pocket**. The triangles of the Delaunay triangulation within a pocket are called **pocket triangles**. The **area of a pocket** is the total area of all pocket triangles. The edge e is called the **door** of the pocket. All MST edges of the cycle are called **border edges**. Those in the interior are called **interior edges**. If e is an MST edge, the pocket will be called an **empty pocket**. Here, e is a border edge and the door at the same time. Empty pockets have area 0.

Note that the door of a pocket is incident to exactly one pocket triangle. If this triangle is obtuse and the door is its hypothenuse, the pocket is called an **obtuse pocket**, otherwise we call it an **acute pocket**.

The **MST-triangles of a pocket** P is the following set of triangles. Every half edge which is part of the MST and incident to a pocket triangle of P generates an MST-triangle. An MST-triangle of a half edge of length l is an equilateral triangle with side length l.

Obviously, both half edges of an interior edge belong to the pocket. On the other hand, only one half edge of a border edge belongs to the pocket. The **MST-area of a pocket** is the sum of the areas of all the MST-triangles. The MST-area of an empty pocket is $\triangle(e)$.

Figure 4 shows a pocket. The door of the pocket is the dashed line. Its border consists of all the edges of the MST connecting the two end points of the pocket. The largest of these edges is denoted by b_1. There are four inner edges. Note that the inner edges have two MST-triangles attached, one for each half edge, whereas the border edges have only one. The area of the pocket consists of the interior of the pocket. Because b_1 is part of the MST whereas the door is not, b_1 is never longer then the door.

Lemma 2. *In a acute pocket with largest border edge b, the difference between MST-area and pocket area is bounded by* $\triangle(|b|)$.

The proof of Lemma 2 is quite complicated. We therefore only give a sketch of it towards the end of this section. Using this lemma, one can prove Lemma 3. Due to lack

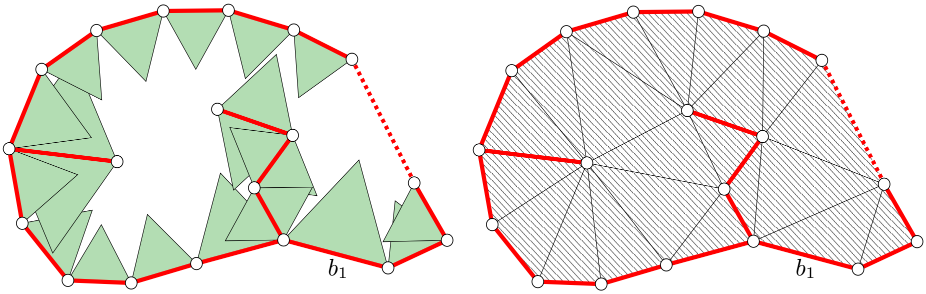

Fig. 4. A pocket with its MST-triangles on the left and the pocket triangles on the right. Note that there are 22 MST-triangles and 21 pocket triangles

to space, also this proof is omitted here. Lemma 3 in turn leads directly to the proof of Theorem 1.

Lemma 3. *Consider a pocket formed by an edge e. Then its MST-area can be bounded by the area of the pocket plus the area of a set of equilateral triangles whose side lengths are bounded by* 1 *and add up to* $|e|$.

Proof of Theorem 1. Consider the pockets whose doors are the edges of the convex hull of S. The sum of the MST-areas of all these pockets is equal to the total MST-area generated by S. Using Lemma 3, we can conclude that the total MST-area is bounded by the area of a so-called sun. A sun is defined by a convex set T from the unit disk, with the additional property that all edges of the convex hull are bounded by 1. The **area of a sun** is the convex hull of T plus, for each edge e of the convex hull of T, the area of an equilateral triangle with side length $|e|$.

Observe that the MST-area of S is $\mu(S) \cdot \sqrt{3}/2$. Hence we just need to prove that the area of a sun is bounded by $6\sqrt{3}/2$, which is exactly the area of a sun produced by a regular hexagon.

A point set T maximizing the area of its sun has all points on the unit circle. This holds since by moving a point towards the unit circle, the area of the sun increases. The area of a sun with all points on the unit circle can be partitioned as indicated in Figure 5(a). Each part consists of the triangle formed by the origin and an edge of the convex hull, plus the corresponding equilateral triangle. The area of each part can be expressed in terms of the angle ρ the first triangle forms at the origin. It is

$$f(\rho) := \sin\left(\frac{\rho}{2}\right)\left(\cos\left(\frac{\rho}{2}\right) + 2 \cdot \sin\left(\frac{\rho}{2}\right)\frac{\sqrt{3}}{2}\right) = \frac{1}{2} sin(\rho) + \sqrt{3} \cdot sin\left(\frac{\rho}{2}\right)^2.$$

Because we assumed that the edges of the convex hull are bounded by 1, the angle ρ must be between 0 and $\frac{\pi}{3}$. Note that $f(\frac{\pi}{3}) = \sqrt{3}/2$. In order to prove that the sun area is maximized by a regular hexagon, observe from Figure 5(b) that $f(\rho)/\rho$, restricted to the range $0 \leq \rho \leq \frac{\pi}{3}$, is maximized for $\rho = \frac{\pi}{3}$. $\square$

In the remainder of this section, we describe the main ideas of the proof of Lemma 2.

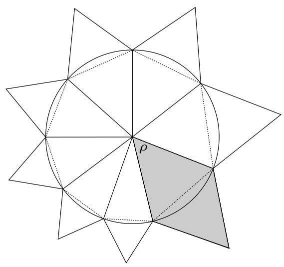

Fig. 5(a). A sun

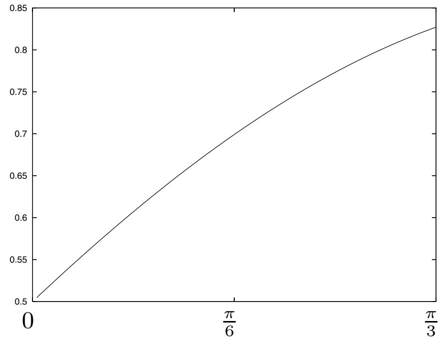

Fig. 5(b). $f(\rho)/\rho = \frac{\frac{1}{2}sin(\rho)+\sqrt{3}sin(\frac{1}{2}\rho)^2}{\rho}$

Lemma 4. *In a pocket, the number of MST-triangles exceeds the number of pocket triangles by one.*

Proof. Let n, e, b, t, be the number of nodes, edges, border edges (not including the door), and pocket triangles of a pocket. Let us first count the MST-triangles. Since the edges of a pocket form a tree, there are $n-1$ edges of the MST involved in the pocket. Each border edge produces one MST-triangle, whereas each inner edges produces two. Therefore their number is $2n-2-b$. Let us now count the pocket triangles. The Descartes-Euler polyhedral formula gives $n+(t+1)-e=2$, where the additional 1 is the face outside the pocket. By double counting the half edges, we obtain $2e = 3t+(b+1)$. Here, the additional 1 stands for the door of the pocket. Solving for t, we get that the number of pocket triangles is $t = 2n-2-b-1$. □

The **extended pocket area (EP-area)** of a pocket is defined as the area of the pocket triangles plus an additional equilateral triangle with side length b, where b is the longest border edge of the pocket. This additional triangle is called **door triangle**. We call the union of the pocket triangles and the door triangle **EP-triangles**. The **net-area** of the pocket is defined as its EP-area minus its MST-area. Using the above definition, Lemma 2 can be rewritten as

Lemma 2 (reformulated) *The net-area of an acute pocket is non-negative.*

Let $V \subseteq S$ be the set of vertices that are part of the pocket, i.e., all the vertices that are inside or at the border of the pocket. Let $G=(V,E)$ be the weighted graph obtained by adding all the edges of the triangulation of the pocket, including the border edges and the door. The weight of the edge $e = uv$, $u, v \in S$, is denoted by $w(e)$ and its value is $|uv|$. Observe that the EP-, MST-, and net-area of a pocket are just a sum of triangle areas. Therefore, using Heron's formula [22]

$$\triangle(a,b,c) = \frac{1}{4}\sqrt{-a^4-b^4-c^4+2\,a^2b^2+2\,a^2c^2+2\,b^2c^2}$$

for the area of a triangle with side lengths a, b, and c, their areas can all be expressed in terms of the weighted planar graph G. What is more, defining EP-, MST-, and net-area in terms of weighted planar graphs allows to define them even if the planar graph does not have an embedding in the plane. The next lemma takes full advantage of this fact.

Lemma 5. *Let G' be a graph obtained from G by setting all weights to the same value. Then the net-area of G' is* 0.

Proof. If all weights of G' are equal, all the triangles involved in the EP-area and the MST-area of G' are equilateral and have the same side length. By Lemma 4, the number of EP-triangles is equal to the number of MST-triangles. Since both the EP-area and the MST-area consist of the same number of congruent triangles, we can conclude that their area is equal. □

We will now define a continuous process that turns G into a graph in which all edges have the same weight. During the process, only the weights of the edges are altered, whereas the combinatorial structure of G remains unchanged. The process is designed in such a way that the net-area of G decreases monotonically. This property, together with Lemma 5, proves that the net-area of G is non-negative.

Let $w_{\min}$ and $w_{\max}$ be the length of the smallest and the largest edge in G. The process will be described by a set of graphs $G(m)$, $m \in \mathbb{R}$, $w_{\min} \leq m \leq w_{\max}$. We start with $G = G(w_{\max})$ and end with $G(w_{\min})$, in which all weights will be $w_{\min}$.

The complete proof is quite involved and therefore deferred to the full version of this paper. In the remainder of the paper, we sketch the proof for a special case. Namely, we assume that all pocket triangles in G are acute. In this case, the process can be described very easily: Let $w(e)$ and $w_m(e)$ be the weight of edge e in G and $G(m)$ respectively. Then $w_m(e) = \min(m, w(e))$. That is, in every stage of the process, all maximal edges are decreased simultaneously until all edges have the same weight.

During this process only maximal edges are decreased. Hence, the ordering of the edges in terms of length remains unchanged during the process. Therefore the MST of G remains valid in all $G(m)$.

It is easy to see that during the process, the area of the pocket triangles of G decrease monotonically. This holds only because we assumed that the pocket triangles are acute.

We need to show that the net-area decreases. Hence, we have to show that in every $G(m)$, the decrease of the pocket area is at least as large as the decrease of the MST-area. To do this, we will partition $G(m)$ into so-called **chains** for which we will prove that their total net-area decreases monotonically.

Consider a graph $G(m)$ for fixed m. Chains are defined in terms of a graph Q. The vertex set of Q is the set of triangles in $G(m)$ plus the door triangle. Each maximal MST edge e of $G(m)$ creates the following set of edges in Q. If we remove e, the MST is divided into two subtrees. Let $R(e)$ be the ring of triangles that separates the two subtrees. For any pair of adjacent triangles in $R(e)$, we add an edge in Q. Note that if e is a border edge, the door triangle is also part of the ring and it is connected with the triangle incident to e and the pocket triangle incident to the door, as shown on the right of Figure 5. This completes the definition of the graph Q.

The **chains** are defined as the connected components of Q. Let Q' be a chain. We can define the area, the MST-area, and the net-area of Q' as follows. The **area of** Q' is equal to the sum of the areas of all triangles of Q'. Concerning the MST-area, note that every MST half edge h of $G(m)$ and its corresponding MST-triangle can be assigned

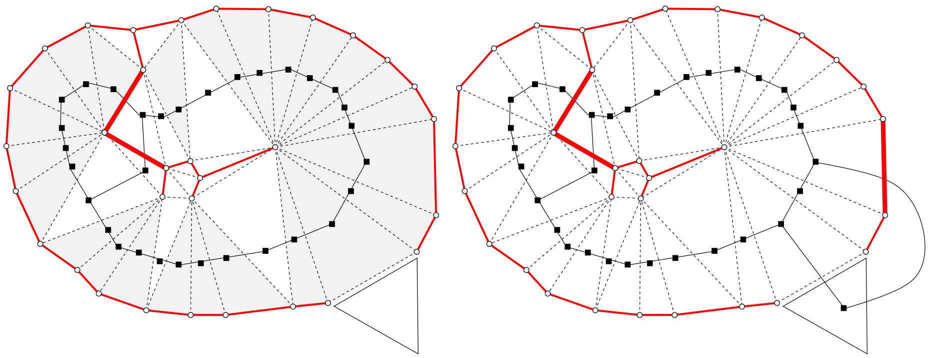

Fig. 6. Two chains in a graph G plus the door triangle. During the process, the chain on the left appears when there are two maximal MST edges, whereas the one on the right appears when there are three maximal MST edges. The maximal edges are the thick MST edges

to a unique chain, namely the one that contains the unique triangle h is incident to. The **MST-area of** Q' is equal to the sum of the areas of all MST-triangles belonging to the chain. The **net-area of** Q' is its area minus its MST-area. Since the net-area of a pocket is equal to the sum of the net-areas of all its chains, all we have to do to complete the proof is to show that the net-area of a chain decreases monotonically.

Remember that in $G(m)$, only the maximal MST edge decrease. Therefore only the MST-triangles of maximal MST edges decrease. In what follows, we will show that the decrease of the area of the chain makes up for the decrease of these **maximal MST-triangles**.

Some chains contain only a single triangle and no maximal MST-triangles. For these chains, it is obvious that the net-area decreases. Let us now look at chains that contain maximal MST-triangles. We need to consider two cases.

In case (i), we assume that the door triangle is not part of the chain. The **border of** Q' is the cycle in G with smallest area that contains all the triangles of Q'. The triangles from Q' that are incident to a border edge are called **border triangles**. On the left of Figure 6, the border triangles are shaded.

Let us now change Q' as follows. Let e be a maximal MST edge belonging to Q'. Assume its two incident triangles are q_1 and q_2. Add two new vertices h_1 and h_2 to Q'. They represent the half edges of e. Then remove the edge q_1q_2 from Q' and add q_1h_1 and q_2h_2. If we do this for all maximal MST edges e, Q' becomes a tree. Let d_i be the number of vertices of Q' with degree i, let n and e be the number of vertices and edges, respectively. From $e = n - 1$ (since Q' is a tree), $n = d_1 + d_2 + d_3$ and $2e = 3d_3 + 2d_2 + d_1$ (by double counting), we get $d_3 = d_1 - 2$.

The vertices of degree three represent equilateral triangles with side length m in $G(m)$. Let D_3 be the set of these triangles. The vertices of degree one represent the maximal MST half edges in $G(m)$. Let D_1 be the set of their corresponding MST-triangles. The area of the triangles in D_1 and D_3 decreases in the same way. Hence, the decrease of all but two triangles from D_1 is made up by the triangles in D_3. To complete the proof, one can show that the decrease of the border triangles of the chain makes up for the decrease of the remaining two maximal MST-triangles. Due to lack of space, we skip this proof here.

For case (ii), assume that the door triangle belongs to Q'. Apply the edge splitting of the previous case to Q'. This time, remove all the leafs adjacent to vertex representing the door triangle. We can do the same analysis as in the previous case to find that if the number of leafs in Q' is d_1, then the number of degree three vertices is $d_3 = d_1 - 2$. But this time, one of the leafs is the door triangle, which is an equilateral triangle with side length m. Hence there are as many half edges as equilateral triangles. Hence, if m decreases, the MST-area of the chain decreases at least as much as the area of the chain.

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