Discrete Event Systems
Solution to Exercise: Petri Nets (Part 1)

1 Structural Properties of Petri Nets and Token Game

Given is the following Petri net $N_1$:

![Petri Net Diagram]

a) What are the input ($\text{Pre}$) and output ($\text{Post}$) sets of transitions $t_5$ and $t_8$? Of place $p_3$?

b) Starting from the initial marking, which transitions are enabled after $t_1$ and $t_2$ have fired?

c) What is the total number of tokens in $N_1$ before and after $t_2$ fired?

d) Play the token game for $N_1$ and construct the reachability graph.

Hint You may denote the states in such a way that the index indicates the places that hold a token in this state, for example $s_0 = (1,0,0,0,1,0,0,0,0,0)$ $\rightarrow$ $s_{1,5}$.

a) The Pre and Post sets of a transition are defined as follows:

- Pre set: $\text{Pre}(t) := \{p \mid (p, t) \in \mathcal{F}\}$
- Post set: $\text{Post}(t) := \{p \mid (t, p) \in \mathcal{F}\}$,
where $F$ is the flow set, i.e., the set of place/transition and transition/place arcs. The Pre and Post sets of a place are defined analogously.

For the Petri net $N_1$ we obtain the following sets:

- $t_5 = \{p_5, p_9\}$, $t_5\bullet = \{p_6\}$
- $t_8 = \{p_8\}$, $t_8\bullet = \{p_{10}, p_5\}$
- $p_3 = \{t_2\}$, $p_3\bullet = \{t_3\}$

b) A transition is enabled if all places in its Pre set contain enough tokens. In the case of $N_1$, which has only unweighted edges, one token per place suffices. When $t_2$ fires, it consumes one token out of each place in the Pre set of $t_2$ and produces one token on each place in the Post set of $t_2$. Hence, the firing of $t_2$ produces one token on place $p_3$ and $p_9$ each, while the token in $p_2$ is consumed.

As a result, $t_5$ is enabled because both $p_9$ and $p_5$ hold one token. However, $t_3$ is not enabled because $p_3$ contains a token but $p_{10}$ does not.

c) Before $t_2$ fires there are two tokens in $N_1$, one in $p_2$ and one in $p_5$. After the firing, there is one token in places $p_3$, $p_9$ and $p_5$, hence 3 tokens in total.

d) A token traverses the upper cycle until $t_2$ fires. Then one token remains on $p_3$ and waits, and another one is produced in $p_9$, which enables transition $t_5$. When $t_5$ consumes the tokens on $p_9$ and $p_5$ and produces a token on $p_6$, this one traverses the lower cycle until $t_3$ is enabled and fired. One token now remains on $p_5$ and waits, another is in $p_{10}$ and enables $t_3$, because there is another token on $p_3$. Then one token traverses the upper cycle again until $t_2$ is enabled, and so on. This Petri net models two alternating processes.

This Petri net is clearly bounded, thus we can construct its reachability tree. Usually the states of Petri nets are denoted by vectors such that the $i$-th position in the vector indicates the number of tokens on place $p_i$ of the Petri net, i.e., the marking of the graph. So, for example, the starting state $\vec{s}_0$ of $N_1$, in which the places $p_1$ and $p_5$ hold one token each, is denoted by $\vec{s}_0 = (1, 0, 0, 0, 1, 0, 0, 0, 0, 0)$.

For better legibility we denote the states in such a way that the index contains the places that hold a token in this state, for example $\vec{s}_0 = (1, 0, 0, 0, 1, 0, 0, 0, 0, 0) \equiv s_{1,5}$.

Then the reachability graph can also be written as,
2 Basic Properties of Petri Nets

Given is the following Petri net $N_2$:

![Petri Net Diagram]

Explain the terms *boundedness* and *deadlock-freeness* using this example, i.e. for which values of $k \in \mathbb{N}$ is the Petri net $N_2$ bounded/unbounded and not deadlock-free?

A Petri net is $k$-bounded, if there is no fire sequence that makes the number of tokens in one place grow larger than $k$. It is obvious that Petri net $N_2$ is 1-bounded if $k \leq 1$. This holds because in the initial state there is only one token in the net, and in the case $k \leq 1$ no transition increases the number of tokens in $N_2$. If $k \geq 2$, the number of tokens in $p_1$ can grow infinitely large by repeatedly firing $t_1$, $t_3$ and $t_4$. So, the Petri net $N_2$ is unbounded for $k \geq 2$.

A Petri net is deadlock free if no fire sequence leads to a state in which no transition is enabled. If $k = 0$, $N_2$ is not deadlock-free. The fire sequence $t_1, t_3, t_4$ causes the only existing token to be consumed and hence, there is no enabled transition any more. For $k \geq 1$, however, no deadlock can occur.
3 Identifying a deadlock

The following Petri net \( N_3 \) describes two linear processes \( (P_{A0/A1/A2} \text{ and } P_{B0/B1/B2}) \) sharing resources \( R_1 \) and \( R_2 \).

In the following, use \( M = (P_{A0}, P_{A1}, P_{A2}, P_{R1}, P_{R2}, P_{B0}, P_{B1}, P_{B2}) \) as marking vector and \( T = (t_{A0}, t_{A1}, t_{A2}, t_{B0}, t_{B1}, t_{B2}) \) as firing vector.

a) Construct the reachability graph of net \( N_3 \) for the given initial marking. Explicit one firing sequence leading to a blocking marking (i.e., to a deadlock). What is the blocking marking you obtain?

b) Express the upstream \( (W^-) \) and downstream \( (W^+) \) incidence matrices and deduce the incidence matrix \( A \). Using \( A \), compute the marking reached after firing \( t_{A0}t_{B0} \) from the initial marking.

c) Using the upstream incidence matrix \( W^- \), how can you prove that the marking obtained in the previous question represents a deadlock state?

d) Suggest a modification to this Petri net which allows the two linear processes \( P_A \) and \( P_B \) to run "as intended in the first place" (i.e. without blocking).
   \textbf{Hint} You don’t need more than one place and a few arcs.
a) There are an infinite number of blocking sequence: any number of cycles $t_{A0}t_{A1}t_{A2}$ and/or $t_{B0}t_{B1}t_{B2}$ terminated by either $t_{A0}t_{B0}$ or $t_{B0}t_{A0}$. It can be read directly from the marking graph below:

![Marking Graph]

$M_0 = (1,0,0,1,1,1,0,0) \quad (0,1,0,0,0,1,0)$

b) From the Petri net structure, we get:

$W^+ = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$, $W^- = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$, and

$A = W^+ - W^- = \begin{bmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & -1 & 1 \\
0 & -1 & 1 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}$

Consider the firing sequence $t_{A0}t_{B0}$. It entails:

$M_{deadlock} = M_0 + A \cdot \begin{bmatrix}
1 \\
0 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
1 \\
0
\end{bmatrix} + \begin{bmatrix}
-1 \\
1 \\
-1 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}$

As expected, we find again the blocking marking from the reachability graph of question a).

c) A deadlock state is a state at which no transition in enabled. Hence, one can use the upstream transition matrix $W^-$ to assess whether or not a marking is blocking. It is the case if and only if the marking vector does not cover (i.e., is bigger or equal to) any column of $W^-$. Otherwise, it implies the transition associated to such column is enabled, and therefore this marking is not blocking.

d) In order to avoid such deadlock, it suffices to forbid both process to run concurrently. This can be solved easily using a semaphore, as illustrated thereafter:
4 From mutual exclusion to starvation

In this exercise, your task is to design a Petri net that models a system in which two processes (A and B) want to access a common exclusive resource, similarly as in Exercise 3. This means that the two processes have to exclude each other mutually from the concurrent access to the resource (e.g. a critical program section). In computer science, this concept is often called a mutex. Here is the desired behavior of the system:

1. One process executes its program.
2. In order to enter the critical section, the mutex value must be 1 (i.e. the mutex is available).
3. If this is the case, the process sets the mutex to 0 and executes its critical section.
4. When it is done, it resets the mutex to 1 and enters an uncritical section.
5. It loops back to start.

a) Propose a Petri net representing the desired behavior.
   Hint Use 5 places and 4 transitions.

b) In this setting, it may happen that one process starves the other. That means one process always uses the resource and the other never enters its critical section (e.g. this may happened when using fixed-priority scheduling). Adapt your model such that process A cannot execute more than two times more than B. This may results in having only one process able to start running from the initial marking.

c) What is the problem of such approach? What would you suggest as a specification for the scheduling of processes A and B?

d) Bonus You can try to express your specification using a Petri Net.
a) For each process we introduce two places \((p_1, p_2, p_3, \text{ and } p_4)\) representing the process within the normal program execution \((p_1, p_2)\) as well as in the critical section \((p_3, p_4)\). For each process, we have a token indicating which section of the program is currently executed. Additionally, we introduce a place \(p_0\) representing the mutex variable. If the mutex variable is 0, then we have a token at \(p_0\). We have to make sure that a process can only enter its critical section if there is a token at the mutex place. The resulting Petri net looks as follows.

Assume that initially, both processes are in an non-critical section (in the Petri net, this is denoted by a token in place \(p_1\) and \(p_2\) respectively). A process can only enter its critical section \((p_3, p_4)\) if there is a token at \(p_0\). In this case, the token is consumed when entering the critical section. A new mutex token at \(p_0\) is not created until the process leaves its critical section. Hence, both processes exclude each other mutually from the concurrent access to the critical section.

This is a classical benefit of Petri nets over other DES models. It models very efficiently the sharing of resources, the concurrency of processes, and so on...

b) In order to avoid starvation of either of the process, one option is to count the number of execution the each of them, or more precisely the difference between them. Assume that at initial state, none has been previously run. According to the specification, we can allow one to the process (say \(A\)) to run twice by creating a "counter-resource" with 2 tokens in the initial marking. Running the process \(A\) consumes one of these tokens. A new token is produced in this place on completion of process \(B\). Doing that symmetrically (for each process) binds the number of executions of each process together...

Not so clear? Okay, just have a look to the net :-)
c) With such mechanism, $A$ will not starve $B$. However, if for some reason, $B$ does not execute anymore, then $A$ will have to stop as well once it has two executions more than $B$. This would a pretty bad design in most cases.

The naive idea would be to say that "if both processes want to access the resource, they get it in turns".

5 Coverability tree and graph

Given is the following Petri net $N_5$, compute its coverability tree and coverability graph. Deduce which are the unbounded places of this net given the initial marking.

Following the procedure from the lecture note, we can construct the following coverability tree:

$$M_0 = \begin{array}{c}
(1,0,0) \\
(1,\omega,0) \\
(1,\omega,\omega) \\
(1,\omega,\omega)
\end{array}$$
One can merge the equivalent node and obtain the coverability graph:

$$M_0 = \begin{pmatrix} (1,0,0) & t_1 & (1,0,0) & t_3 & (1,0,0) \\ t_1, t_2 & t_1, t_2, t_3 \end{pmatrix}$$

It follows that for this net with initial marking $(1,0,0)$, places $p_2$ and $p_3$ are unbounded.

## 6 Reachability Analysis for Petri Nets

In the lecture we presented an algorithm to perform a reachability analysis on Petri nets.

a) Why is it not possible with a reachability algorithm to determine in general, whether a given state in a Petri net is reachable or not? To which extend the notion of coverability helps?

b) Consider the Petri net $N_2$ from exercise 2. Is the state $s = (p_1 = 101, p_2 = 99, p_3 = 4)$ reachable from the initial state $s_0 = (1,0,0)$ if $k = 2$? Prove your answer, either by giving a suitable firing sequence, or a justification for non-reachability.

**Hint** Start with incidence matrix of $N_2$ and express the necessary condition (presented in the lecture) for the reachability of the marking $s$ from $s_0$. Eventually, discuss whether or not the marking is reachable.

a) Petri nets may possess infinite reachability graphs, e.g. $N_2$ with $k \geq 2$. If a marking is actually reachable in such a Petri net, the reachability check will eventually terminate. But if it is not reachable, the algorithm may not be able to determine reachability with absolute certainty (cf. halting problem).

Constructing a coverability tree or graph is guaranteed to terminate. It can be used to prove that a given marking is not reachable, in the case where the marking you are interested is not covered by any marking in the coverability tree/graph. However, this is not a sufficient to prove reachability in the general case: a marking may be covered by the coverability graph, and yet not being reachable.

b) We determine the incidence matrix of the Petri net as explained in the lecture.

$$A = \begin{pmatrix} -1 & 1 & 0 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

We are interested in whether the state $\vec{s} = (101,99,4)$ is reachable from the initial state $s_0 = (1,0,0)$. If the equation system $A \cdot \vec{f} = \vec{s} - \vec{s}_0$ has no solution, we know for sure that the state $\vec{s}$ is not reachable from $s_0$. "Unfortunately",

$$\begin{pmatrix} -1 & 1 & 0 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 99 \\ 4 \end{pmatrix}$$

is satisfiable. Using linear algebra, the solutions to this system can be computed (here, $f_1 = Q, f_2 = Q - 306, f_3 = 207, f_4 = 203$, for any $Q \in \mathbb{N}$). If $\vec{s}$ is reachable from $s_0$, the firing sequence will be of this form. However, there is no guarantee that it is actually feasible for the net! Ultimately, one has to look at the net and propose a suitable firing sequence (although the solution to the previous system of equations gives us the "shape" of the firing sequence we are looking for).

So, to prove that $\vec{s}$ is reachable from $s_0$, we have to give a firing sequence through which we get from $s_0$ to $\vec{s}$. Considering the Petri net, we can see that starting from $s_0$ – the number of tokens in $p_1$ increases by one after firing the sequence $t_1, t_3, t_4$. Repeating this for 203 times yields the state $(204,0,0)$. Firing $t_1$ for 103 more times, followed by firing $t_3$ for four times finally yields state $\vec{s}$.