Crash course – Verification of Finite Automata
Binary Decision Diagrams

Exercise session 10
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Equivalence of representations

Sets

• Set algebra
• $\cup$, $\cap$, $\neg$

Boolean functions/
Characteristic functions

$\psi_E = 1$
$\psi_A = f$
$\psi_B = g$
$\psi_{A \cap B} = f \cdot g$
Equivalence of representations

Sets
- Set algebra
- $\cup, \cap, \neg$

Boolean functions/Characteristic functions
- Boole algebra
- $+, \cdot, \neg$

Binary encoding:
$\sigma: X \rightarrow \Psi$
$\rightarrow$ Map a state to a binary expression
$\rightarrow$ Bijective mapping!
Equivalence of representations

- Set algebra
- $\cup$, $\cap$, $\neg$

Binary encoding:
\[
\sigma: X \to \Psi
\]

$\Rightarrow$ Map a state to a binary expression
$\Rightarrow$ Bijective mapping!

Boolean functions/Characteristic functions

- Boole algebra
- $+$, $\cdot$, $\neg$

Truth tables:
\[
\begin{align*}
\psi_E &= 1 \\
\psi_A &= f \\
\psi_B &= g \\
\psi_{A \cap B} &= f \cdot g
\end{align*}
\]
Equivalence of representations

Sets

• A
• \( s \in A \)

Example:
\( \sigma(s) = x_1 \overline{x_0} = (1,0) \) and \( \psi_A = x_1 + x_0 \)
\( \rightarrow s \models \psi_A \) ?

Boolean functions/
Characteristic functions

\( \psi_E = 1 \)
\( \psi_A = f \)
\( \psi_B = g \)
\( \psi_{A \cap B} = f \cdot g \)

\( \psi_A \)
\( \psi_A(\sigma(s)) = 1 \)
\( \sigma(s) \models \psi_A \)

or just \( s \models \psi_A \)

Reads “s satisfies \( \psi_A \)”
Binary Decision Diagrams

Based on the Boole-Shannon decomposition:

\[
f = \overline{x} \cdot f \bigg|_{x=0} + x \cdot f \bigg|_{x=1}
\]

Boolean function of \(n\) and \((n - 1)\) variables

→ For a given order of variable, the decomposition is unique!
→ Hence the uniqueness of R(reduced)O(rdered)BDD.

**Reminder:**
In practice, simplicity of BDD depends strongly on the order.

![Good vs. Bad ordering](image-url)
Binary Decision Diagrams: an example

\[ f: x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} \overline{x_2} + \overline{x_2} \overline{x_3} \]

Fall \( x_1 = 0 \)

\[ f|_{x_1=0} : x_2 + \overline{x_2} \overline{x_3} \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]

Fall \( x_1 = 0 \)
\[ f|_{x_1=0} : x_2 + \overline{x_2} \overline{x_3} \]

Fall \( x_2 = 0 \)
\[ f|_{x_1=0, x_2=0} : \overline{x_3} \]
Binary Decision Diagrams: an example

\[ f: x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]

Fall \( x_1 = 0 \)
\[ f|_{x_1=0}: x_2 + \overline{x_2} x_3 \]

Fall \( x_2 = 0 \)
\[ f|_{x_1=0, x_2=0}: \overline{x_3} \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]

Fall: \( x_1 = 0 \)

\[ f |_{x_1=0} : x_2 + \overline{x_2} x_3 \]

Fall: \( x_2 = 0 \)

\[ f |_{x_1=0, x_2=0} : \overline{x_3} \]

Fall: \( x_2 = 1 \)

\[ f |_{x_1=0, x_2=1} : 1 \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]

Fall \( x_1 = 0 \)
\[ f_{x_1=0} : x_2 + \overline{x_2} x_3 \]
Fall \( x_2 = 0 \)
\[ f_{x_1=0, x_2=0} : \overline{x_3} \]
Fall \( x_2 = 1 \)
\[ f_{x_1=0, x_2=1} : 1 \]

Fall \( x_1 = 1 \)
\[ f_{x_1=1} : 1 \]
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Binary Decision Diagrams

Your turn!

Slides online on my webpage:
http://people.ee.ethz.ch/~jacobr/
Ex1: Sets Representation

“Each state is either a nominal or an error state or both".

\[ \Rightarrow \quad N \cup E = X \quad \Leftrightarrow \quad \psi_N + \psi_E = 1 \]
Ex1: Sets Representation

“If a state is in the overflow set, it is not a nominal state”.

\[ N \cap O = \emptyset \iff \psi_N \cdot \psi_O = 0 \]

But note it is not necessarily true!!
Although you would like it to be...
Ex1: Sets Representation

Describe $Q_1$, the set of error states which are not an overflow, in term of sets and characteristic functions.

$$Q_1 = E \setminus O \iff \psi_{Q_1} = \psi_E \cdot \overline{\psi_O}$$
Ex1: Sets Representation

- Describe Q2, satisfying "O ⇒ E", i.e., the set of state for which this property holds, in term of sets and characteristic functions.

Is that correct?
Ex1: Sets Representation

- Describe Q2, satisfying "O ⇒ E", i.e., the set of state for which this property holds, in term of sets and characteristic functions.

Is that correct? No!

→ What if a state is not in O?

Property is always true!
Ex1: Sets Representation

- Describe Q2, satisfying "O ⇒ E", i.e., the set of states for which this property holds, in terms of sets and characteristic functions.

$$Q_2 = (O \cap E) \cup \overline{O} = (O \cup \overline{O}) \cap (E \cup \overline{O})$$

$$= X \cap (E \cup \overline{O})$$

$$= E \cup \overline{O} \quad \Leftrightarrow \quad \psi Q_2 = \psi_E + \overline{\psi_O}$$
Ex2.1 Verification using BDDs

\[ a) \quad f_2 : y = x_1 + x_2 + x_3 + x_1 + x_2 + x_3 + x_1 + x_2 + x_3 \]
Ex2.1 Verification using BDDs

\[ f_1 : (x_1\overline{x_2} + x_1x_3 + \overline{x_2}x_3 + \overline{x_1}x_2x_3) \]

**Fall** \( x_1 = 0 \)

\[ y|x_1=0 = \overline{x_2}x_3 + x_2\overline{x_3} \]

**Fall** \( x_2 = 0 \)

\[ y|x_1=0,x_2=0 = x_3 \]

**Fall** \( x_2 = 1 \)

\[ y|x_1=0,x_2=1 = \overline{x_3} \]

**Fall** \( x_1 = 1 \)

\[ y|x_1=1 = \overline{x_2} + x_3 + \overline{x_2}x_3 \]

**Fall** \( x_2 = 0 \)

\[ y|x_1=1,x_2=0 = 1 \]

**Fall** \( x_2 = 1 \)

\[ y|x_1=1,x_2=1 = x_3 \]
Ex2.1 Verication using BDDs

\[ f_2 : y = \overline{x_1 + x_2 + x_3} + \overline{x_1 + x_2 + x_3} + \overline{x_1 + x_2 + x_3} \]

**Fall** \( x_1 = 0 \)
\[ y|_{x_1=0} = \overline{x_2 + x_3 + x_2 + x_3} \]

**Fall** \( x_2 = 0 \)
\[ y|_{x_1=0,x_2=0} = \overline{x_3 + 1 + x_3} = x_3 \]

**Fall** \( x_2 = 1 \)
\[ y|_{x_1=0,x_2=1} = \overline{1 + x_3} = x_3 \]

**Fall** \( x_1 = 1 \)
\[ y|_{x_1=1} = \overline{1 + 1 + x_2 + x_3} = \overline{x_2 + x_3} \]

**Fall** \( x_2 = 0 \)
\[ y|_{x_1=1,x_2=0} = 1 \]

**Fall** \( x_2 = 1 \)
\[ y|_{x_1=1,x_2=1} = x_3 \]
Ex2.2 BDDs with respect to different orderings

\[ g(x_1, x_2, y_1, y_2) = (x_1 \equiv y_1) \cdot (x_2 \equiv y_2), \quad \Pi : x_1 < x_2 < y_1 < y_2 \]

a) \[ g = x_1 \{ x_2[y_1(y_2) + \overline{y_1}(0)] + \overline{x_2}[y_1(\overline{y_2}) + \overline{y_1}(0)] \} + \overline{x_1} \{ x_2[y_1(0) + \overline{y_1}(y_2)] + \overline{x_2}[y_1(0) + \overline{y_1}(y_2)] \} \]

b)
Ex2.2 BDDs with respect to different orderings

\[ g(x_1, x_2, y_1, y_2) = (x_1 \equiv y_1) \cdot (x_2 \equiv y_2), \quad \Pi' : x_1 < y_1 < x_2 < y_2 \]

c) \[ g = x_1 \{ y_1 [ x_2 (y_2) + \overline{x_2} (\overline{y_2}) ] + \overline{y_1} [0] \} + \overline{x_1} \{ y_1 [0] + \overline{y_1} [ x_2 (y_2) + \overline{x_2} (\overline{y_2}) ] \} \]

Better ordering:
6 vs. 9 nodes
About the lecture exercise feedback
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See you next week!