Crash course – Verification of Finite Automata
Binary Decision Diagrams

Exercise session 6
Xiaoxi He
Equivalence of representations

- Set algebra
- $\cup$, $\cap$, $\neg$

Boolean functions/Characteristic functions
- Boole algebra
- $+$, $\cdot$, $\neg$

$\psi_E = 1$
$\psi_A = f$
$\psi_B = g$
$\psi_{A \cap B} = f \cdot g$
Equivalence of representations

Sets
- Set algebra
- \( \cup, \cap, \neg \)

Boolean functions/Characteristic functions
- Boole algebra
- \( +, \cdot, \neg \)

Binary encoding:
\[ \sigma: X \rightarrow \Psi \]
- Map a state to a binary expression
- Bijective mapping!
Equivalence of representations

- Set algebra
- \( \cup, \cap, \neg \)

Binary encoding:
\[ \sigma: X \rightarrow \Psi \]
- Map a state to a binary expression
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Boolean functions/Characteristic functions
- Boole algebra
- \(+, \cdot, \neg\)
Equivalence of representations

Sets

- A
- \( s \in A \) (proposition)

Example:

\[
\sigma(s) = x_1 \overline{x_0} = (1,0) \quad \text{and} \quad \psi_A = x_1 + x_0
\]

\[ \rightarrow s \vDash \psi_A ? \]

Boolean functions/Characteristic functions

- \( \psi_E = 1 \)
- \( \psi_A = f \)
- \( \psi_B = g \)
- \( \psi_{A \cap B} = f \cdot g \)

\( \sigma(\cdot) \)

Reads “s satisfies \( \psi_A \)”
Binary Decision Diagrams

Based on the Boole-Shannon decomposition:

\[ f = \overline{x} \cdot f \big|_{x=0} + x \cdot f \big|_{x=1} \]

→ For a given order of variable, the decomposition is unique!
→ Hence the uniqueness of \(R\text{(reduced)}O\text{(rdered)}\)BDD.

Reminder:
In practice, simplicity of BDD depends strongly on the order.

Good vs. Bad ordering
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} \, x_2 + \overline{x_2} \, x_3 \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]

Fall \( x_1 = 0 \)

\[ f|_{x_1=0} : x_2 + \overline{x_2} x_3 \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1}x_2 + \overline{x_2}x_3 \]

Fall \( x_1 = 0 \)
\[ f|_{x_1=0} : x_2 + \overline{x_2}x_3 \]
Fall \( x_2 = 0 \)
\[ f|_{x_1=0, x_2=0} : \overline{x_3} \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} \cdot x_2 + \overline{x_2} \cdot x_3 \]

Fall \( x_1 = 0 \)
\[ f_{x_1=0} : x_2 + \overline{x_2} \cdot x_3 \]

Fall \( x_2 = 0 \)
\[ f_{x_1=0, x_2=0} : \overline{x_3} \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x}_1 \ x_2 + \overline{x}_2 \overline{x}_3 \]

Fall \( x_1 = 0 \)
\[ f_{|x_1=0} : x_2 + \overline{x}_2 \overline{x}_3 \]
Fall \( x_2 = 0 \)
\[ f_{|x_1=0, x_2=0} : \overline{x}_3 \]
Fall \( x_2 = 1 \)
\[ f_{|x_1=0, x_2=1} : 1 \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]

Fall \( x_1 = 0 \)

\[ f|_{x_1=0} : x_2 + \overline{x_2} x_3 \]

Fall \( x_2 = 0 \)

\[ f|_{x_1=0, x_2=0} : \overline{x_3} \]

Fall \( x_2 = 1 \)

\[ f|_{x_1=0, x_2=1} : 1 \]

Fall \( x_1 = 1 \)

\[ f|_{x_1=1} : 1 \]
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Binary Decision Diagrams

Your turn!
Ex1: Sets Representation

“Each state is either a nominal or an error state or both".

\[ N \cup E = X \iff \psi_N + \psi_E = 1 \]
Ex1: Sets Representation

“If a state is in the overflow set, it is not a nominal state”.

⇒ \[ N \cap O = \emptyset \iff \psi_N \cdot \psi_O = 0 \]

But note it is not necessarily true!!
Although you would like it to be...
Ex1: Sets Representation

Describe $Q_1$, the set of error states which are not an overflow, in term of sets and characteristic functions.

$Q_1 = E \setminus O \Leftrightarrow \psi_{Q_1} = \psi_E \cdot \overline{\psi_O}$
Ex1: Sets Representation

- Describe Q2, satisfying "O ⇒ E", i.e., the set of state for which this property holds, in term of sets and characteristic functions.

Is that correct?
Ex1: Sets Representation

- Describe $Q_2$, satisfying "$O \Rightarrow E$", i.e., the set of state for which this property holds, in term of sets and characteristic functions.

Is that correct? No!

→ What if a state is not in $O$?

Property is always true!
Ex1: Sets Representation

- Describe Q2, satisfying "O ⇒ E", i.e., the set of state for which this property holds, in term of sets and characteristic functions.

⇒ 

\[ Q_2 = (O \cap E) \cup \overline{O} = (O \cup \overline{O}) \cap (E \cup \overline{O}) = X \cap (E \cup \overline{O}) = E \cup \overline{O} \] 

⇔ \[ \psi_{Q_2} = \psi_E + \psi_O \]

Is that correct? No!

→ What if a state is not in O?

Property is always true!
Ex2.1 Verification using BDDs

\[
\text{a) } f_2 : y = x_1 + x_2 + x_3 + x_1 + x_2 + x_3 + \overline{x_1} + x_2 + x_3
\]
Ex2.1 Verication using BDDs

\[ f_1 : (x_1 \overline{x_2} + x_1 x_3 + \overline{x_2} x_3 + \overline{x_1} x_2 \overline{x_3}) \]

\text{Fall } x_1 = 0
\[ y|_{x_1=0} = \overline{x_2} x_3 + x_2 \overline{x_3} \]

\text{Fall } x_2 = 0
\[ y|_{x_1=0,x_2=0} = x_3 \]

\text{Fall } x_2 = 1
\[ y|_{x_1=0,x_2=1} = \overline{x_3} \]

\text{Fall } x_1 = 1
\[ y|_{x_1=1} = \overline{x_2} + x_3 + \overline{x_2} x_3 \]

\text{Fall } x_2 = 0
\[ y|_{x_1=1,x_2=0} = 1 \]

\text{Fall } x_2 = 1
\[ y|_{x_1=1,x_2=1} = x_3 \]
Ex2.1 Verication using BDDs

\[ f_2 : y = \overline{x_1 + x_2 + x_3 + x_1 + \overline{x_2 + x_3} + \overline{x_1 + x_2 + x_3}} \]

**Fall** \( x_1 = 0 \)
\[ y_{|x_1=0} = \overline{x_2 + x_3 + x_2 + x_3} \]

**Fall** \( x_2 = 0 \)
\[ y_{|x_1=0,x_2=0} = \overline{x_3 + 1} + x_3 = x_3 \]

**Fall** \( x_2 = 1 \)
\[ y_{|x_1=0,x_2=1} = \overline{1 + x_3} = x_3 \]

**Fall** \( x_1 = 1 \)
\[ y_{|x_1=1} = \overline{1 + \overline{1} + x_2 + x_3} = x_2 + x_3 \]

**Fall** \( x_2 = 0 \)
\[ y_{|x_1=1,x_2=0} = 1 \]

**Fall** \( x_2 = 1 \)
\[ y_{|x_1=1,x_2=1} = x_3 \]
Ex2.2 BDDs with respect to different orderings

\[ g(x_1, x_2, y_1, y_2) = (x_1 \equiv y_1) \cdot (x_2 \equiv y_2), \quad \Pi : x_1 < x_2 < y_1 < y_2 \]

a) \[ g = x_1 \{ x_2 [y_1(y_2) + \overline{y_1}(0)] + \overline{x_2} [y_1(\overline{y_2}) + \overline{y_1}(0)] \} \]
   \[ + \overline{x_1} \{ x_2 [y_1(0) + \overline{y_1}(y_2)] + \overline{x_2} [y_1(0) + \overline{y_1}(\overline{y_2})] \} \]

b)
Ex2.2 BDDs with respect to different orderings

\[ g(x_1, x_2, y_1, y_2) = (x_1 \equiv y_1) \cdot (x_2 \equiv y_2), \quad \Pi' : x_1 < y_1 < x_2 < y_2 \]

\( c) \quad g = x_1 \{ y_1 [x_2(y_2) + \overline{x_2(y_2)}] + \overline{y_1}[0] \}
\[ + \overline{x_1} \{ y_1[0] + \overline{y_1}[x_2(y_2) + \overline{x_2(y_2)}] \} \]

Better ordering:
6 vs. 9 nodes