Crash course – Verification of Finite Automata
Binary Decision Diagrams

Exercise session - 23.11.2017
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Equivalence of representations

\[ \psi_E = 1 \]
\[ \psi_A = f \]
\[ \psi_B = g \]
\[ \psi_{A \cap B} = f \cdot g \]

\[ A \cap B \]

Sets
- Set algebra
- \( \cup, \cap, \neg \)

Boolean functions/
Characteristic functions
- Boole algebra
- \( +, \cdot, \neg \)
Equivalence of representations

Set algebra
- $\cup$, $\cap$, $\neg$

Boolean functions/Characteristic functions
- Boole algebra
- $+$, $\cdot$, $\neg$

Binary encoding:
$\sigma: X \to \Psi$

- Map a state to a binary expression
- Bijective mapping!
Equivalence of representations

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Sets
- Set algebra
- \( \cup, \cap, \neg \)

Boolean functions/Characteristic functions
- Boole algebra
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Binary encoding:
\[ \sigma: X \to \Psi \]
- Map a state to a binary expression
- Bijective mapping!
Equivalence of representations

Sets

• A
• \( s \in A \)

Example:
\[ \sigma(s) = x_1 \overline{x_0} = (1,0) \]
\[ \text{and } \psi_A = x_1 + x_0 \]
\[ \rightarrow s \models \psi_A ? \]

Boolean functions/Characteristic functions

\[ \psi_E = 1 \]
\[ \psi_A = f \]
\[ \psi_B = g \]
\[ \psi_{A \cap B} = f \cdot g \]

\( \sigma(\cdot) \)

Reads “s satisfies \( \psi_A \)”
Binary Decision Diagrams

Based on the Boole-Shannon decomposition:

\[ f = \overline{x} \cdot f \bigg|_{x=0} + x \cdot f \bigg|_{x=1} \]

Boolean function of \( n \) and \((n - 1)\) variables

→ For a given order of variable, the decomposition is unique!

→ Hence the uniqueness of O(ordered)BDD.

Reminder:
In practice, simplicity of BDD depends strongly on the order.

Good vs. Bad ordering
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x}_1 x_2 + \overline{x}_2 x_3 \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]

Fall \( x_1 = 0 \)
\[ f|_{x_1=0} : x_2 + \overline{x_2} x_3 \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x}_1 \, x_2 + \overline{x}_2 \, x_3 \]

Fall \( x_1 = 0 \)
\[ f|_{x_1=0} : x_2 + \overline{x}_2 \, x_3 \]
Fall \( x_2 = 0 \)
\[ f|_{x_1=0, x_2=0} : \overline{x}_3 \]

\[ f|_{x_1=0, x_2=0} \]

0 1

\[ \begin{array}{c|c|c}
0 & 0 & 1 \\
1 & & \\
\end{array} \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]

Fall \( x_1 = 0 \)

\[ f|_{x_1=0} : x_2 + \overline{x_2} x_3 \]

Fall \( x_2 = 0 \)

\[ f|_{x_1=0, x_2=0} : \overline{x_3} \]
Binary Decision Diagrams: an example

\[ f: x_1 + \overline{x}_1 x_2 + \overline{x}_2 \overline{x}_3 \]

Fall \( x_1 = 0 \)
\[ f|_{x_1=0}: x_2 + \overline{x}_2 \overline{x}_3 \]
Fall \( x_2 = 0 \)
\[ f|_{x_1=0, x_2=0}: \overline{x}_3 \]
Fall \( x_2 = 1 \)
\[ f|_{x_1=0, x_2=1}: 1 \]
Binary Decision Diagrams: an example

\[ f : x_1 + \overline{x_1} x_2 + \overline{x_2} x_3 \]

Fall \( x_1 = 0 \)

\[ f_{|x_1=0} : x_2 + \overline{x_2} x_3 \]

Fall \( x_2 = 0 \)

\[ f_{|x_1=0, x_2=0} : \overline{x_3} \]

Fall \( x_2 = 1 \)

\[ f_{|x_1=0, x_2=1} : 1 \]

Fall \( x_1 = 1 \)

\[ f_{|x_1=1} : 1 \]
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Binary Decision Diagrams

Your turn!

Slides online on my webpage:
http://people.ee.ethz.ch/~jacobr/
Ex1: Sets Representation

“Each state is either a nominal or an error state”.

\[
N \cup E = X \iff \psi_N + \psi_E = 1
\]
Ex1: Sets Representation

“If a state is in the overflow set, it is not a nominal state”.

\[ N \cap O = \emptyset \quad \iff \quad \psi_N \cdot \psi_O = 0 \]

But note it is not necessarily true!!
Although you would like it to be...
Ex1: Sets Representation

Describe $Q_1$, the set of error states which are not an overflow, in term of sets and characteristic functions.

$\Rightarrow \quad Q_1 = E \setminus O \quad \iff \quad \psi_{Q_1} = \psi_E \cdot \overline{\psi_O}$
Ex1: Sets Representation

- Describe Q2, satisfying "O ⇒ E", i.e., the set of state for which this property holds, in term of sets and characteristic functions.

Is that correct?
Ex1: Sets Representation

- Describe Q2, satisfying "O ⇒ E", i.e., the set of state for which this property holds, in term of sets and characteristic functions.

Is that correct? No!

→ What if a state is not in O?

Property is always true!
Ex1: Sets Representation

- Describe Q2, satisfying "O ⇒ E", i.e., the set of state for which this property holds, in term of sets and characteristic functions.

\[ Q_2 = (O \cap E) \cup \overline{O} = (O \cup \overline{O}) \cap (E \cup \overline{O}) = X \cap (E \cup \overline{O}) = E \cup \overline{O} \]

\[ \iff \psi_{Q_2} = \psi_E + \psi_O \]

Is that correct? No!

→ What if a state is not in O?

Property is always true!
Ex2.1 Verification using BDDs

\[ f_2 : y = \overline{x_1 + x_2 + x_3} + \overline{x_1 + x_2 + \overline{x_3}} + \overline{x_1 + x_2 + x_3} \]
Ex2.1 Verification using BDDs

\[ f_1 : (x_1\overline{x}_2 + x_1x_3 + \overline{x}_2 x_3 + \overline{x}_1 x_2 \overline{x}_3) \]

**Fall \( x_1 = 0 \)**
\[ y|_{x_1=0} = \overline{x}_2 x_3 + x_2 \overline{x}_3 \]

**Fall \( x_2 = 0 \)**
\[ y|_{x_1=0,x_2=0} = x_3 \]

**Fall \( x_2 = 1 \)**
\[ y|_{x_1=0,x_2=1} = \overline{x}_3 \]

**Fall \( x_1 = 1 \)**
\[ y|_{x_1=1} = \overline{x}_2 + x_3 + \overline{x}_2 x_3 \]

**Fall \( x_2 = 0 \)**
\[ y|_{x_1=1,x_2=0} = 1 \]

**Fall \( x_2 = 1 \)**
\[ y|_{x_1=1,x_2=1} = x_3 \]
Ex2.1 Verication using BDDs

\[ f_2 : y = \overline{x_1 + x_2 + x_3} + \overline{x_1 + x_2 + x_3} + \overline{x_1 + x_2 + x_3} \]

\[ \text{Fall } x_1 = 0 \]
\[ y_{|x_1=0} = \overline{x_2 + x_3 + x_2 + x_3} \]

\[ \text{Fall } x_2 = 0 \]
\[ y_{|x_1=0,x_2=0} = x_3 + 1 + x_3 = x_3 \]

\[ \text{Fall } x_2 = 1 \]
\[ y_{|x_1=0,x_2=1} = \overline{1 + x_3} = x_3 \]

\[ \text{Fall } x_1 = 1 \]
\[ y_{|x_1=1} = \overline{1 + x_1 + x_2} + x_3 = x_2 + x_3 \]

\[ \text{Fall } x_2 = 0 \]
\[ y_{|x_1=1,x_2=0} = 1 \]

\[ \text{Fall } x_2 = 1 \]
\[ y_{|x_1=1,x_2=1} = x_3 \]
Ex2.2 BDDs with respect to different orderings

\[ g(x_1, x_2, y_1, y_2) = (x_1 \equiv y_1) \cdot (x_2 \equiv y_2), \quad \Pi : x_1 < x_2 < y_1 < y_2 \]

a) \[ g = x_1 \{ x_2[y_1(y_2) + \overline{y_1}(0)] + \overline{x_2}[y_1(\overline{y_2}) + \overline{y_1}(0)] \} + \overline{x_1} \{ x_2[y_1(0) + \overline{y_1}(y_2)] + \overline{x_2}[y_1(0) + \overline{y_1}(\overline{y_2})] \} \]

b) 

Diagram of BDD with nodes labeled accordingly.
Ex2.2 BDDs with respect to different orderings

\[ g(x_1, x_2, y_1, y_2) = (x_1 \equiv y_1) \cdot (x_2 \equiv y_2), \quad \Pi' : x_1 < y_1 < x_2 < y_2 \]

c) \[ g = x_1 \{y_1[x_2(y_2) + \overline{x_2(y_2)}] + \overline{y_1}[0] \} + \overline{x_1} \{y_1[0] + \overline{y_1}[x_2(y_2) + \overline{x_2(y_2)}] \} \]

Better ordering:
6 vs. 9 nodes
About the lecture exercise feedback
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See you next week!