Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Distributed Computing

HS 2024

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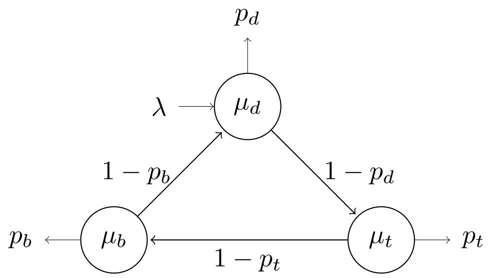
Discrete Event Systems

Solution to Exercise Sheet 8

1 Queuing Networks

a)



b) Note that the waiting queue of the dispatcher is an $M/M/1$ queue with arrival rate λ_d and service rate μ_d. Assuming stability ($\mu_d > \lambda_d$), Burke's Theorem tells us that the time between two departures from the waiting system of the dispatcher is exponentially distributed with parameter λ_d.

By the same argument, the times between two departures from the waiting systems of the technician and the Beep agent are exponentially distributed with parameters λ_t and λ_b, respectively.

At the waiting queue of the dispatcher, new customers arrive with rate λ and returning customers are forwarded by the Beep agent with rate $\lambda_b(1 - p_b)$. Thus, the arrival rates $\lambda_d, \lambda_t, \lambda_b$ at the waiting queues can be calculated as follows:

$$\begin{aligned}
\lambda_d &= \lambda + \lambda_b(1 - p_b) \\
\lambda_t &= \lambda_d(1 - p_d) \\
\lambda_b &= \lambda_t(1 - p_t)
\end{aligned}$$

Solving this system of linear equations gives:

$$\lambda_d = \frac{\lambda}{1 - (1 - p_d)(1 - p_t)(1 - p_b)}$$

$$\lambda_t = \frac{(1 - p_d)\lambda}{1 - (1 - p_d)(1 - p_t)(1 - p_b)}$$

$$\lambda_b = \frac{(1 - p_d)(1 - p_t)\lambda}{1 - (1 - p_d)(1 - p_t)(1 - p_b)}$$

c) The waiting queue of the technician is an $M/M/1$ queue with utility $\rho_t = \frac{\lambda_t}{\mu_t}$. Hence, the average waiting time is given by $W_t = \frac{\rho_t}{\mu_t - \lambda_t}$.

d) We apply the given values to the equations for λ_d, λ_t and λ_b and obtain:

$$\lambda_d = 10, \qquad \lambda_t = 25/3, \qquad \lambda_b = 20/3.$$

We confirm staility for each node:

$$\lambda_d = 10 < 20 = \mu_d, \qquad \lambda_t = 25/3 < 10 = \mu_t, \qquad \lambda_b = 20/3 < 10 = \mu_b.$$

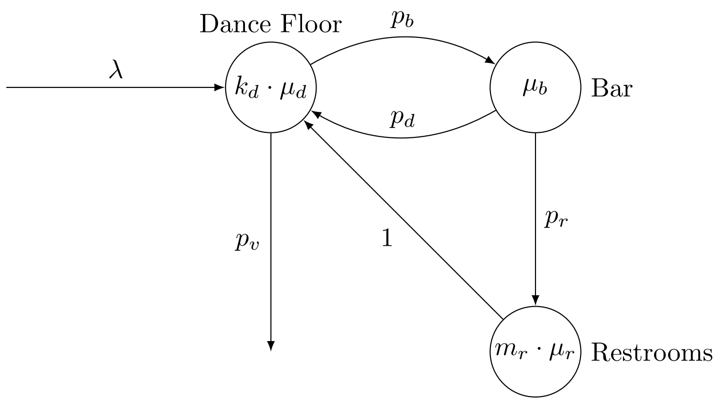
According to Theorem 4.21, there are $N = \frac{\lambda}{\mu - \lambda}$ jobs in an $M/M/1$ queue in expectation. Thus, by linearity of expectation, the expected number of customers in the system is given by

$$N = \frac{\lambda_d}{\mu_d - \lambda_d} + \frac{\lambda_t}{\mu_t - \lambda_t} + \frac{\lambda_b}{\mu_b - \lambda_b} = 8.$$

Applying Little's formula to the entire system gives $T = \frac{N}{\lambda} = \frac{8}{5}$ hours.

2 A Night at the DISCO

a) As a queuing network, the DISCO can be modeled as follows.



The bar can be modeled as an $M/M/1$ queue, while the restroom can be modeled as an $M/M/m$ queue, where m denotes the number of restrooms. The service rate at the bar is a fixed μ_b, and the service rate at the restroom is $m_r \cdot \mu_r$, where $m_r = \min\{k_r, m\}$, k_r is the number of people at the restrooms and m is the number of toilets (cubicles).

As there is no waiting line in front of the dance floor and the maximum number of people on the dance floor is not bounded, we can model the dance floor as an $M/M/\infty$ queue. Note that the service rate is $k_d \cdot \mu_d$, where k_d denotes the number of people currently on the dance floor.

Note that for the following exercise, it does not matter what types of queue we have, because Burke's Theorem holds for any stable $M/M/m$ queue, even for $m = \infty$.

b) With the same argument as in Exercise 1, we obtain the following system of linear equations:

$$\begin{aligned}
\lambda_d &= \lambda + \lambda_b \cdot p_d + \lambda_r \\
\lambda_b &= \lambda_d \cdot p_b \\
\lambda_r &= \lambda_b \cdot p_r \,.
\end{aligned}$$

Solving for λ_d yields

$$\lambda_d = \frac{\lambda}{1 - p_b \cdot p_d - p_b \cdot p_r} \,.$$

c) We need to ensure that $\lambda_r/(m \cdot \mu_r) < 1$. Counting in hours, we have that $\lambda_r = 90$ and $\mu_r = 12$, which yields that m must be at least 8 (since there is no such thing as half a toilet).

d) This is incorrect for two reasons: First of all, the parameters λ, μ_d, μ_b of the exponential distribution do not represent a time interval, but a rate, i.e. the number of people arriving (or being served) per time interval. Their inverses $\frac{1}{\lambda}, \frac{1}{\mu_d}, \frac{1}{\mu_b}$ represent the expected time between two users arriving at the DISCO or the expected time a user spends at the bar or the restrooms, respectively.

Moreover, we cannot calculate the expected time until a user goes to the restrooms by simply summing up λ, μ_d, μ_b (or their inverses), because with probability p_v, the visitor does not go to the bar at all, and even if he does, there is still a positive probability p_d that he does not go to the restrooms afterwards.