Discrete Event Systems
Solution to Exercise Sheet 11

1 Comparison of Finite Automata

Here are two simple finite automata:

For each, we have a one bit encoding for the states ($x_A$ and $x_B$), one binary output ($y_A$ and $y_B$), and one common binary input ($u$). We want to verify whether or not these two automata are equivalent. This can be done through the following steps:

a) Express the characteristic function of the transition relation for both automaton, $\psi_r(x, x', u)$.

b) Express the joint transition function, $\psi_f$.
   Reminder: $\psi_f(x_A, x'_A, x_B, x'_B) = (\exists u: \psi_A(x_A, x'_A, u) \cdot \psi_B(x_B, x'_B, u))$.

c) Express the characteristic function of the reachable states, $\psi_X(x_A, x_B)$.

d) Express the characteristic function of the reachable output, $\psi_Y(y_A, y_B)$.

e) Are the two automata equivalent? Hint: Evaluate, for example, $\psi_Y(0, 1)$.

a) $\psi_A(x_A, x'_A, u) = x_A x'_A u + x_A x'_A u + x_A x'_A u + x_A x'_A u$

$\psi_B(x_B, x'_B, u) = x_B x'_B u + x_B x'_B u + x_B x'_B u + x_B x'_B u$

b) $\psi_f(x_A, x'_A, x_B, x'_B) = (x_A x'_A + x_A x'_A) \cdot (x_B x'_B + x_B x'_B) +$

$(x_A x'_A + x_A x'_A) \cdot (x_B x'_B + x_B x'_B)$

$= x_A x'_A x_B x'_B + x_A x'_A x_B x'_B + x_A x'_A x_B x'_B + x_A x'_A x_B x'_B +$

$x_A x'_A x_B x'_B + x_A x'_A x_B x'_B + x_A x'_A x_B x'_B + x_A x'_A x_B x'_B$

c) Computation of the reachable states is performed incrementally. Starts with the initial state of the system $\psi_X_0(x_A, x_B) = x_A x_B$ and then add the successors until reaching a fix-point,
\( \psi_{X_1}(x'_A, x'_B) = \psi_{X_0}(x'_A, x'_B) + (\exists (x_A, x_B) : \psi_{X_0}(x_A, x_B) \cdot \psi_f(x_A, x'_A, x_B, x'_B)) \)

\[ = x_A x'_B + x'_A x'_B + x'_A x_B \]

\( \psi_{X_2}(x'_A, x'_B) = \psi_{X_1}(x'_A, x'_B) + x'_A x'_B + x'_A x_B + x_A x'_B \)

\( \psi_{X_3}(x'_A, x'_B) = \psi_{X_1}(x'_A, x'_B) + x'_A x'_B + x'_A x_B + x_A x'_B = \psi_{X_2} \rightarrow \text{the fix-point is reached!} \)

\[ \Rightarrow \psi_X(x_A, x_B) = x_A x_B + x_A x_B + x_A x_B + x_A x_B \]

d) Here you first need to express the output function of each automaton, that is the feasible combinations of states and outputs,

\( \psi_{gA} = x_A y_A + x_A y_A \) and \( \psi_{gB} = x_B y_B + x_B y_B \)

Then the reachable outputs are the combination of the reachable states and the outputs functions, that is,

\( \psi_Y(y_A, y_B) = (\exists (x_A, x_B) : \psi_X \cdot \psi_{gA} \cdot \psi_{gB}) \)

\[ = y_A y_B + y_A y_B + y_A y_B + y_A y_B \]

e) From the reachable output function, we see that these automata are not equivalent. Indeed, there exists a reachable output admissible \((\psi_Y((y_A, y_B) = (0, 1)) = 1)\) for which \(y_A \neq y_B\).

Another way of saying looking at it: \( \psi_Y \cdot (y_A \neq y_B) \neq 0, \)

where \((y_A \neq y_B) = y_A y_B + y_A y_B \).
2 Temporal Logic

a) We consider the following automaton. The property $a$ is true on the colored states (0 and 3).

For each of the following CTL formula, list all the states for which it holds true.

1. $EF\ a$
2. $EG\ a$
3. $EX\ AX\ a$
4. $EF\ (a\ AND\ EX\ NOT(a))$

(i) $Q = \{0, 1, 2, 3\}$
(ii) $Q = \{0, 3\}$
(iii) $(AX\ a)$ holds for $\{2, 3\}$, thus $Q = \{1, 2\}$
(iv) $(a\ AND\ EX\ NOT(a))$ is true for states where $a$ is true and there exists a direct successor for which it is not. Only state 0 satisfy this (from it you can transition to 1, where $a$ does not hold). Moreover, state 0 is reachable for all states in this automaton (“from all states there exists a path going through 0 at some point”) Hence $Q = \{0, 1, 2, 3\}$

b) Given the transition function $\psi_f(q, q')$ and the characteristic function $\psi_Z(q)$ for a set $Z$, write a small pseudo-code which returns the characteristic function of $\psi_{AF\ Z}(q)$. It can be expressed as symbolic boolean functions, like $x_A x_B + x_A x_B$.

Hint: To do this, simply use the classic boolean operators $\AND$, $\OR$, $\NOT$ and $\not=$. You can also use the operator $\PRE(Q, f)$, which returns the predecessor of the set $Q$ by the transition function $f$. That is,

$$\PRE(Q, f) = \{q' : \exists q, \psi_f(q', q) \cdot \psi_Q(q) = 1\}$$

Hint: It can be useful to reformulate $AF\ Z$ as another CTL formula.

Here, the trick is to remember that $AF\ Z \equiv \NOT(EG\ NOT(Z))$. Hence, one can compute the function for $EG\ NOT(Z)$ quite easily (following the procedure given in the lecture) and take the negation in the end. A possible pseudo-code doing this is the following.

Require: $\psi_Z, \psi_f$

current = $\NOT(\psi_Z)$; $\triangleright$ Equivalence in term of sets:
next = current $\AND\ \psi_{\PRE(current,f)}$; $\triangleright\ X_0$
while next $\not=$ current do
    current = next;
    next = current $\AND\ \psi_{\PRE(current,f)}$; $\triangleright\ X_1 = X_0 \cap Pre(X_0, f)$
end while
return $\psi_{AF\ Z} = \NOT(\psi_Q)$; $\triangleright\ X_f = EG\ NOT(Z)$