Let's prove that

$$L = \{ \text{"}x + y = z\text{"} \mid x, y, z \in \{0,1\}^* \wedge \text{satisfy the equation} \}$$

is <u>not</u> context-free.

Let's pick $s = \text{"}1^{p+1} = 1^p + 10^p\text{"}$ as string.
Clearly, $|s| \geq p$.

Also, it should be clear that $s \in L$.
Take this three instantiations as an example:

1. $(p=1)$ $\quad \overset{3}{11} = \overset{1}{1} + \overset{2}{10}$
2. $(p=2)$ $\quad \overset{7}{111} = \overset{3}{11} + \overset{4}{100}$
3. $(p=3)$ $\quad \overset{15}{1111} = \overset{7}{111} + \overset{8}{1000}$

Let's think now about all the ways we could split s into $uvxyz$ s.t. all the conditions of the tandem pumping lemma apply.

Let's keep it mind 2 things:

1) $|vxy| \leq p$; and
2) $uv^i x y^i z \in L, \forall i \geq 0$.

Clearly, we see that if the "vxy" part is entirely contained in the Left-Hand Side (LHS) or Right-Hand Side (RHS) of the equation, then the equation <u>cannot</u> be valid anymore as we start repeating the v and y parts.

So that means, the vxy part must be "in the middle". Also we need to keep in mind that we can only have one "=".

So we end up with split like this

$$\underbrace{11\ldots\overset{v}{\boxed{1}}}_{p+1} \overset{x}{=} \underbrace{\overset{y}{\boxed{11}}\ldots 1}_{p} + \underbrace{10000}_{p+1}$$

Given $|vxy| \leq p$ and the x part being necessarily "=". We have that the v and y parts will have a length of $(p-1)/2$

$$\underbrace{11\ldots\underbrace{\boxed{\overbrace{1}^{(p-1)/2}}}_{v}}_{p+1}\;\underbrace{\boxed{\overbrace{=}^{1}}}_{x}\;\underbrace{\underbrace{\boxed{\overbrace{11}^{(p-1)/2}}}_{y}\ldots 1}_{p} + \underbrace{10\ldots0}_{p+1}$$

Depending on the value of p, we see that v and y, either:

- will have the same length, if p is odd
- " " $\neq$ " , " " even

In either case though, the equation won't be satisfied. The intuition being that v repeats in the least significant bit part of the LHS, while the y part repeats in the most significant bits part of the RHS.

Taking two concrete examples:

$$(p=3) \qquad 111\underbrace{\boxed{1}}_{v}\;\underbrace{\boxed{=}}_{x}\;\underbrace{\boxed{1}}_{y}11 + 1000$$

$$v^2xy^2 \rightarrow \underbrace{11111}_{31} = \underbrace{1111}_{15} + \underbrace{1000}_{8}.$$

The equation does not match anymore.

$(p=4)$ $$111\overbrace{\boxed{11}}^{z}\overbrace{\boxed{=}}^{x}\overbrace{\boxed{1}}^{y}1111 + 10000$$

(or vice-versa, with $|z|=1$ and $|y|=2$)

$$z^2xy^2 \to \underbrace{111\,1111}_{127} = \underbrace{11111}_{31} + \underbrace{10000}_{16}$$

In all cases, we see that repeating v and y lead to strings that do <u>NOT</u> belong to the language anymore.