Discrete Event Systems
Verification of Finite Automata (Part 1)

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Most materials from Lothar Thiele and Romain Jacob
What are finite automata useful for?
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- Digital circuits
- Protocols (e.g. BGP)
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- **Specification**
  - Digital circuits
  - Protocols (e.g. BGP)

- **Simulation**
  - Anything specified with automata
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- Digital circuits
- Protocols (e.g. BGP)

**simulation**
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**synthesis of software or hardware**
- Hardware components
- Network configurations
What are finite automata useful for?

- **specification**
  - Digital circuits
  - Protocols (e.g. BGP)

- **simulation**
  - Anything specified with automata

- **verification**

- **synthesis of software or hardware**
  - Digital circuits
  - Network configurations
Verification of Finite Automata

Questions:

• Does the system specification model the desired behavior correctly?
• Do implementation and specification describe the same behavior?
• Can the system enter an undesired (or dangerous) state?
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• Simulation (sometimes also called validation or testing)
  • Unless the simulation is exhaustive, i.e., all possible input sequences are tested, the result is not trustworthy.
  • In general, simulation can only show the presence of errors but not the absence (correctness).
Verification of Finite Automata

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Possible solutions:
  • Simulation (sometimes also called validation or testing)
    • Unless the simulation is exhaustive, i.e., all possible input sequences are tested, the result is not trustworthy.
    • In general, simulation can only show the presence of errors but not the absence (correctness).
  • Formal analysis (sometimes also called verification)
    • Formal (unambiguous) proof of correctness.
Verification of Finite Automata

- Due to the finite number of states, proving properties of a finite state machine can be done by enumeration.
- As computer systems have finite memory, properties of processors (and embedded systems in general) could be shown in principle.
Verification of Finite Automata

- Due to the finite number of states, proving properties of a finite state machine can be done by enumeration.
- As computer systems have finite memory, properties of processors (and embedded systems in general) could be shown in principle.
- But is enumeration a reasonable approach in practice?

<table>
<thead>
<tr>
<th>memory</th>
<th>number of states</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 Bit</td>
<td>256</td>
</tr>
<tr>
<td>32 Bit</td>
<td>$4 \times 10^9$</td>
</tr>
<tr>
<td>1KBit</td>
<td>$10^{300}$</td>
</tr>
<tr>
<td>1MBit</td>
<td>$10^{300 000}$</td>
</tr>
<tr>
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# atoms in the universe is about $10^{82}$
Verification of Finite Automata

- There have been **major breakthroughs** in recent years on the verification of finite automata with very large state spaces. Prominent methods are based on
  - transformation to a Boolean Satisfiability (SAT) problem (not covered in this course) and
  - symbolic model checking via binary decision diagrams (covered in this course).
Verification of Finite Automata

• There have been **major breakthroughs** in recent years on the verification of finite automata with very large state spaces. Prominent methods are based on
  • transformation to a Boolean Satisfiability (SAT) problem (not covered in this course) and
  • symbolic model checking via binary decision diagrams (covered in this course).

• **Symbolic model checking** is a method of verifying temporal properties of finite (and sometimes infinite) state systems that relies on a symbolic representation of sets, typically as Binary Decision Diagrams (BDD’s).

• **Verification** is used in industry for proving the correctness of complex digital circuits (control, arithmetic units, cache coherence), safety-critical software and embedded systems (traffic control, train systems, security protocols).
Verification Scenarios

Example

\[ y = (x_1 + x_2) \cdot x_3 \]

Comparison of specification and implementation

- Reference system → data structure
- System under test → data structure
- Comparison
Verification Scenarios

Example

\[ y = (x_1 + x_2) \cdot x_3 \]

“The device can always be switched off.”

Comparison of specification and implementation

- reference system → data structure
- system under test → data structure
- comparison

Proving properties

- property
- fixed-point calculation

- system under test → data structure
Efficient state representation

- Set of states as Boolean function
- Binary Decision Diagram representation

Computing reachability

- Leverage efficient state representation
- Explore successor sets of states

Proving properties

- Temporal logic (CTL)
- Encoding as reachability problem
This week

Efficient state representation
- Set of states as Boolean function
- Binary Decision Diagram representation

Computing reachability
- Leverage efficient state representation
- Explore successor sets of states

Proving properties
- Temporal logic (CTL)
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Binary Decision Diagrams (BDD)

• Concept
  • Data structure that allows to represent Boolean functions.
  • The representation is unique for a given ordering of variables. If the ordering of variables is fixed, we call it an ordered BDD (OBDD).

• Structure
  • BDDs contain “decision nodes” which are labeled with variable names.
  • Edges are labeled with input values.
  • Leaves are labeled with output values.

\[ f = x_1 + x_2 + x_3 \]
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Basic concept of verification using BDDs

• BDDs represent Boolean functions.

• Therefore, they can be used to describe sets of states and transformation relations.

• Due to the unique representation of Boolean functions, reduced ordered BDDs (ROBDD) can be used to proof equivalence between Boolean functions or between sets of states.

• BDDs can easily and efficiently be manipulated.
Decomposition

BDDs are based on the Boole-Shannon-Decomposition:

\[ f = \bar{x} \cdot f\big|_{x=0} + x \cdot f\big|_{x=1} \]

A Boolean function has two co-factors for each variable, one for each evaluation

- \( f\big|_{x=0} \): remaining function for \( x = 0 \)
- \( f\big|_{x=1} \): remaining function for \( x = 1 \)
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\[ f = x_1 + x_2 + x_3 = x_1 \cdot 1 + \overline{x_1} \cdot (x_2 + x_3) \]

\[ f|_{x_1=1} \quad f|_{x_1=0} \]
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\[ = x_2 + x_2 \cdot x_3 \]
Boole-Shannon Decomposition Example

\[ f(a, b, c) = \overline{a} \cdot (b + c) + \overline{b} \cdot c \]

Ordering: \( a \rightarrow b \rightarrow c \)
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Does variable order matter?
Variable Order

- If we fix the ordering of variables, BDDs are called OBBDs (Ordered Binary Decision Diagrams).
- The ordering is essential for the size of a BDD.

\[ f = (a \cdot b) + (c \cdot d) + e \]
Calculating with BDDs

- **SIMPLIFY**: Given BDD for \( f \), determine simplified BDD for \( f \).
  - Eliminate redundant nodes.
    - Merge equivalent leaves (\( 0 \) and \( 1 \))
    - Merge isomorphic nodes, i.e., nodes that represent the same Boolean function.
  - A BDD that can not be further simplified is called a reduced BDD.
  
  A reduced OBDD (also denoted as ROBDD) is a unique representation of a given Boolean function.
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- **SIMPPLY**: Given BDD for \( f \), determine simplified BDD for \( f \).
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\text{Merge leaves}
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$$f = \overline{a} \cdot (b + c) + \overline{b} \cdot c$$
Calculating with BDDs

- **RESTRICT**: Given BDD for $f$, determine BDD for $f|_{x=k}$.
  
  - Delete all edges that represent $x = \bar{k}$;
  
  - For every pair of edges $(a - x, x - b)$ include a new edge $(a - b)$ and remove the old ones;
  
  - Remove all nodes that represent $x$. 
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![Diagram of BDDs before and after restrict operation]
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f = \overline{a} \cdot (b + c) + \overline{b} \cdot c
\]
Calculating with BDDs

• **APPLY**: Given BDDs for \( f \) and \( g \), determine a BDD for \( f \diamond g \) for some operation \( \diamond \).
  
  • Combine the two BDDs recursively based on the following relation:

  \[
  f \diamond g = \overline{x} \cdot (f \mid_{x=0} \diamond g \mid_{x=0}) + x \cdot (f \mid_{x=1} \diamond g \mid_{x=1})
  \]

  ![Diagram](attachment:image.png)

• Boolean functions can be converted to BDDs step by step using **APPLY**.
Calculating with BDDs

• Quantifiers are constructed by **APPLY** and **RESTRICT**:

\[
(\exists x : f) \iff (f \mid_{x=0} + f \mid_{x=1})
\]

\[
(\forall x : f) \iff (f \mid_{x=0} \cdot f \mid_{x=1})
\]

\[
(\exists x_1, x_2 : f) \iff (\exists x_1 (\exists x_2 : f))
\]

\[
(\forall x_1, x_2 : f) \iff (\forall x_1 (\forall x_2 : f))
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\[f(a, b) = \bar{a} \cdot b\]
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f(a, b) = \bar{a} \cdot b \\
g(a) = \exists b : f(a, b)
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Calculating with BDDs

- Quantifiers are constructed by **APPLY** and **RESTRICT**:

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(\exists x : f) \iff (f \mid_{x=0} + f \mid_{x=1}) \\
(\forall x : f) \iff (f \mid_{x=0} \cdot f \mid_{x=1})
\]

\[
f(a, b) = \bar{a} \cdot b \quad g(a) = \exists b : f(a, b) = \bar{a} \quad h(a) = \forall b : f(a, b) = 0
\]
Comparison using BDDs

• Boolean (combinatorial) circuits: Compare specification and implementation, or compare two implementations.

• Method:
  • Representation of the two systems in ROBDDs, e.g., by applying the `APPLY` operator repeatedly.
  • Compare the structures of the ROBDDs.

• Example:
Sets and Relations

• Representation of a subset $A \subseteq E$:
Sets and Relations

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  • Binary encoding $\sigma(e)$ of all elements $e \in E$
Sets and Relations

- Representation of a subset $A \subseteq E$:
  - Binary encoding $\sigma(e)$ of all elements $e \in E$
  - Subset $A$ is represented by $a \in A \iff \psi_A(\sigma(a))$

\[
\begin{align*}
\sigma(e_1) &= (0, 1, 0) \\
\sigma(e_2) &= (0, 0, 0) \\
\psi_A(\sigma(e_1)) &= 0 \\
\psi_A(\sigma(e_2)) &= 1
\end{align*}
\]
Sets and Relations

- Representation of a subset $A \subseteq E$:
  - Binary encoding $\sigma(e)$ of all elements $e \in E$
  - Subset $A$ is represented by $a \in A \iff \psi_A(\sigma(a))$
  - Stepwise construction of the BDD corresponding to some subsets.

\[
\begin{align*}
c \in A \cap B & \iff \psi_A(\sigma(c)) \cdot \psi_B(\sigma(c)) \\
c \in A \cup B & \iff \psi_A(\sigma(c)) + \psi_B(\sigma(c)) \\
c \in A \setminus B & \iff \psi_A(\sigma(c)) \cdot \psi_B(\sigma(c)) \\
c \in E \setminus A & \iff \psi_A(\sigma(c))
\end{align*}
\]
Sets and Relations

- Example:

\[
\forall e \in E : \sigma(e) = (x_1, x_0) \\
\sigma(e_0) = (0, 0) \quad \sigma(e_1) = (0, 1) \quad \sigma(e_2) = (1, 0) \quad \sigma(e_3) = (1, 1) \\
\psi_A = x_0 \oplus x_1
\]

\[A = ?\]
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\[A = \{e_1, e_2\}\]
Sets and Relations

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Capitals? $\psi_A(x_1, x_0) = ?$

European cities? $\psi_B(x_1, x_0) = ?$

European capitals? $\psi_c(x_1, x_0) = ?$
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Capital? 

European cities?

European capitals?

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Capitals? $\psi_A(x_1, x_0) = ?$  $\psi_A(x_1, x_0) = x_1$

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<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Beijing</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Paris</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Capitals? \quad \psi_A(x_1, x_0) = ? \quad \psi_A(x_1, x_0) = x_1

European cities? \quad \psi_B(x_1, x_0) = ? \quad \psi_B(x_1, x_0) = \bar{x}_0 \cdot \bar{x}_1 + x_0 \cdot x_1

European capitals? \quad \psi_c(x_1, x_0) = ? \quad C = A \cap B \quad \psi_c(x_1, x_0) = x_0 \cdot x_1
Selecting a “good” encoding is both important and difficult

For a state space encoded with $N$ bits

Represent up to $2^N$ states

In previous example

Subset $A$ of all capitals is represented by $\psi_A = x_1$

- No need to iterate through all capitals to verify that some property holds (e.g. “All capitals have a parliament.”)
- We can use the (compact) representation of the set.
Selecting a “good” encoding is both important and difficult

For a state space encoded with $N$ bits

Represent up to $2^N$ states

In previous example

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- No need to iterate through all capitals to verify that some property holds (e.g. “All capitals have a parliament.”)
- We can use the (compact) representation of the set.

But...

Selecting a good encoding — Representing state efficiently is difficult in practice.

- It is one challenge of ML: How to efficiently encode the inputs?
Efficient state representation

- Set of states as Boolean function
- Binary Decision Diagram representation

Computing reachability

- Leverage efficient state representation
- Explore successor sets of states

Proving properties

- Temporal logic (CTL)
- Encoding as reachability problem
Sets and Relations using BDDs

• Representation of a relation $R \subseteq A \times B$
  • Binary encoding $\sigma(a)$, $\sigma(b)$ of all elements $a \in A$, $b \in B$
  • Representation of $R$

  $$(a, b) \in R \iff \psi_R(\sigma(a), \sigma(b))$$

characteristic function of the relation $R$
Sets and Relations using BDDs

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  • Binary encoding $\sigma(a), \sigma(b)$ of all elements $a \in A, b \in B$
  • Representation of $R$

\[(a, b) \in R \iff \psi_R(\sigma(a), \sigma(b))\]

characteristic function of the relation $R$

• Example:

\[\psi_\delta(\sigma(q), \sigma(q')) = \psi_\delta(q, q')\]

describe state transitions return 1 if there is a transition $q \to q'$, 0 otherwise

\[\psi_\delta(q_0, q_1) = 1\]
\[\psi_\delta(q_0, q_3) = 0\]
Reachability of States

- Problem: Is a state \( q \in Q \) reachable by a sequence of state transitions?
- Method:
  - Represent set of states and the transformation relation as ROBDDs.
  - Use these representations to transform from one set of states to another. Set \( Q_i \) corresponds to the set of states reachable after \( i \) transitions.
  - Iterate the transformation until a fixed-point is reached, i.e., until the set of states does not change anymore (steady-state).
- Example:

\[
Q_0 = \{ q_0 \} \quad Q_1 = \{ q_0, q_1 \} \quad Q_2 = \{ q_0, q_1, q_2 \} \quad Q_3 = \{ q_0, q_1, q_2 \}
\]
Drawing state-diagrams is not feasible in general.
Drawing state-diagrams is not feasible in general.

1. Work with sets of states
2. Use characteristic functions to represent sets of states
3. Use ROBDDs to encode characteristic functions
Reachability of States

- Transformation of sets of states:
  - Determine the set of all direct successor states of a given set of states $Q$ by means of the transformation function $\delta$:

$$Q' = \text{Suc}(Q, \delta) = \{q' \mid \exists q : \psi_Q(q) \cdot \psi_\delta(q, q')\}$$

- Characteristic function of current state set $Q$
- Transition function $q \rightarrow q'$

Set of successor states: $Q' = \text{Suc}(Q, \delta) = \{q_1\}$

Characteristic function of current state set $Q$

Transition function $q \rightarrow q'$

$Q_0 = \{q_0\}$

$Q' = \text{Suc}(Q_0, \delta) = \{q_1\}$
Reachability of States

• Transformation of sets of states:
  • Determine the set of all direct successor states of a given set of states \( Q \) by means of the transformation function \( \delta \):

\[
Q' = \text{Suc}(Q, \delta) = \{ q' \mid \exists q : \psi_Q(q) \cdot \psi_\delta(q, q') \}
\]

Set of successor states: \( Q' = \text{Suc}(Q, \delta) = \{ q' \mid \exists q : \psi_Q(q) \cdot \psi_\delta(q, q') \} \)

Characteristic function of current state set \( Q \)
Transition function \( q \to q' \)

states with at least one outgoing transition

\[
\psi_\delta(q, q')
\]

\[
q \quad \delta \quad q'
\]

states with at least one incoming transition

set of all states

set of all states
Reachability of States

• Transformation of sets of states:
  • Determine the set of all direct successor states of a given set of states $Q$ by means of the transformation function $\delta$:

$$Q' = Succ(Q, \delta) = \{ q' | \exists q : \psi_Q(q) \cdot \psi_\delta(q, q') \}$$

Set of successor states: $Q' = Succ(Q, \delta) = \{ q' | \exists q : \psi_Q(q) \cdot \psi_\delta(q, q') \}$

Characteristic function of current state set $Q$

Transition function $q \rightarrow q'$

states with at least one outgoing transition

states with at least one incoming transition

$\psi_Q(q)$

set of all states

$\psi_\delta(q, q')$

$q \delta q'$

set of all states
Reachability of States

- Transformation of sets of states:
  - Determine the set of all direct successor states of a given set of states $Q$ by means of the transformation function $\delta$:

$$Q' = Suc(Q, \delta) = \{q' \mid \exists q : \psi_Q(q) \cdot \psi_\delta(q, q')\}$$

**Set of successor states: $Q' = Suc(Q, \delta) = \{q' \mid \exists q : \psi_Q(q) \cdot \psi_\delta(q, q')\}$**

$$\psi_Q(q) \cdot \psi_\delta(q, q')$$

Characteristic function of current state set $Q$

Transition function $q \rightarrow q'$
Reachability of States

• Transformation of sets of states:
  • Determine the set of all direct successor states of a given set of states $Q$ by means of the transformation function $\delta$:

  
  \[
  \text{Set of successor states: } Q' = \text{Suc}(Q, \delta) = \{q' : \exists q : \psi_Q(q) \cdot \psi_\delta(q, q')\}
  \]

  Efficient to compute with ROBDDs

  \[
  h(q, q') = \psi_Q(q) \cdot \psi_\delta(q, q') \\
  \psi_{Q'}(q') = (\exists q : h(q, q'))
  \]
Reachability of States

- Fixed-point iteration
  - Start with the initial state, then determine the set of states that can be reached in one or more steps.

\[
\begin{align*}
Q_0 &= \{q_0\} \\
Q_{i+1} &= Q_i \cup \text{Suc}(Q_i, \delta) \\
\psi_{Q_{i+1}}(q') &= \psi_{Q_i}(q') + (\exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q, q'))
\end{align*}
\]

\(q'\) is already in \(Q_i\)

There is a state \(q\) in \(Q_i\) with transition \(q \rightarrow q'\)

\[Q_0 = \{q_0\}\]
\[Q' = \text{Suc}(Q_0, \delta) = \{q_1\}\]
\[Q_1 = Q_0 \cup \text{Suc}(Q_0, \delta) = \{q_0, q_1\}\]
Reachability of States

• Fixed-point iteration
  • Start with the initial state, then determine the set of states that can be reached in one or more steps.

\[ Q_0 = \{q_0\} \]

\[ Q_{i+1} = Q_i \cup \text{Suc}(Q_i, \delta) \quad \text{until} \quad Q_{i+1} = Q_i \]

Characteristic function of next set of reached states

\[ \psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + \left( \exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q, q') \right) \]

- \( q' \) is already in \( Q_i \)
- There is a state \( q \) in \( Q_i \) with transition \( q \rightarrow q' \)

• Due to the finite number of states, the fixed-point exists and is reached in a finite number of steps (at most the diameter of the state diagram).
• Determine whether the fixed-point is reached or not can be done by comparing the ROBDDs of the current set of reachable states.
Reachability of States - Example

<table>
<thead>
<tr>
<th>$\sigma(q)$</th>
<th>$x_1$</th>
<th>$x_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_0$</td>
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State encoding
$(x_1, x_0) = \sigma(q)$
Reachability of States - Example

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State encoding \((x_1, x_0) = \sigma(q)\)

Transition relation encoding \( \psi_\delta(q, q') \)

- As a Boolean function

\[
\psi_\delta(q, q') = x_0' \cdot (x_0 \cdot (x_1 + x_1') + x_1 \cdot x_1') + x_0 \cdot x_0' \cdot x_1'
\]

entries where \( \psi_\delta(q, q') = 1 \) only

<table>
<thead>
<tr>
<th>( x_1 )</th>
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\( q_0 \to q_1 \)

\( q_2 \to q_2 \)
Reachability of States - Example

State encoding
\( (x_1, x_0) = \sigma(q) \)

Transition relation encoding \( \psi_\delta(q, q') \)

As a Boolean function
\[
\psi_\delta(q, q') = \overline{x_0} \cdot (x_0 \cdot (x_1 + x'_1) + x_1 \cdot x'_1) + \overline{x_0} \cdot x'_0 \cdot \overline{x_1}
\]

Compute reachable states:
\[
\psi_{Q_{i+1}}(q') = \psi_{Q_{i}}(q') + (\exists q : \psi_{Q_{i}}(q) \cdot \psi_\delta(q, q'))
\]
Reachability of States - Example

### State encoding

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State encoding $(x_1, x_0) = \sigma(q)$

### Transition relation encoding

$\psi_\delta(q, q')$

As a Boolean function

$$\psi_\delta(q, q') = \overline{x_0} \cdot (x_0 \cdot (x_1 + x_1') + x_1 \cdot x_1') + \overline{x_0} \cdot x_0' \cdot \overline{x_1}$$

### Compute reachable states:

$\psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + (\exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q, q'))$

$Q_0 = \{q_0\}$

$\psi_{Q_0}(q) = \overline{x_1} \cdot \overline{x_0}$
Reachability of States - Example

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State encoding \((x_1, x_0) = \sigma(q)\)

Transition relation encoding \(\psi_\delta(q, q')\)

As a Boolean function
\[
\psi_\delta(q, q') = \overline{x_0'} \cdot (x_0 \cdot (x_1 + x_1') + x_1 \cdot x_1') + \overline{x_0} \cdot x_0' \cdot \overline{x_1'}
\]

Compute reachable states:
\[
\psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + (\exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q, q'))
\]

\[
\psi_{Q_0}(q) = \overline{x_1} \cdot \overline{x_0}
\]

\[
\psi_{Q_1}(q') = \overline{x_1'} \cdot \overline{x_0} + (\exists q : \overline{x_1} \cdot \overline{x_0} \cdot \psi_\delta(q, q'))
\]

\(Q_1 = Q_0 \cup \{q_1\} = \{q_0, q_1\}\)
Reachability of States - Example

<table>
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<tr>
<th>( \sigma(q) )</th>
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State encoding \( (x_1, x_0) = \sigma(q) \)

Transition relation encoding \( \psi_\delta(q, q') \)

As a Boolean function
\[
\psi_\delta(q, q') = x'_0 \cdot (x_0 \cdot (x_1 + x'_1) + x_1 \cdot x'_1) + x_0 \cdot x'_0 \cdot x'_1
\]

Compute reachable states:
\[
\psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + (\exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q, q'))
\]

\[
\psi_{Q_0}(q) = \overline{x_1} \cdot \overline{x_0}
\]
\[
\psi_{Q_1}(q') = \overline{x'_1} \cdot \overline{x'_0} + (\exists q : \overline{x_1} \cdot \overline{x_0} \cdot \psi_\delta(q, q'))
\]

From BDDs and quantifiers:
\[
\exists x : f = f \bigg|_{x=0} + f \bigg|_{x=1}
\]
The only non-zero term is for \( x_0=0, x_1=0 \) (see next slide)
Reachability of States – Example (BDD Calculation)

\[ \psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + (\exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q, q')) \]

\[ \psi_{Q_0}(q) = \overline{x_1} \cdot \overline{x_0} \]

**Eq. 1:** \[ \psi_{Q_1}(q') = \overline{x_1} \cdot \overline{x_0} + (\exists q : \overline{x_1} \cdot \overline{x_0} \cdot \psi_\delta(q, q')) \]

\[ \exists x_0 : f \quad f \bigg|_{x_0 = 1} = \overline{x_1} \cdot 0 \cdot (\overline{x_0'} \cdot (1 \cdot (x_1 + x_1') + x_1 \cdot x_1') + 0 \cdot x_0' \cdot x_1') = 0 \]

\[ f \big|_{x_0 = 0} = \overline{x_1} \cdot 1 \cdot (\overline{x_0'} \cdot (0 \cdot (x_1 + x_1') + x_1 \cdot x_1') + 1 \cdot x_0' \cdot x_1') = \overline{x_1} \cdot (\overline{x_0'} \cdot (x_1 \cdot x_1') + x_0' \cdot x_1') \]

\[ \exists x_1 : f \bigg|_{x_0 = 0} \quad f \bigg|_{x_0 = 0, x_1 = 1} = 0 \cdot (\overline{x_0'} \cdot (1 \cdot x_1') + x_0' \cdot x_1') = 0 \]

\[ f \bigg|_{x_0 = 0, x_1 = 0} = 1 \cdot (\overline{x_0'} \cdot (0 \cdot x_1') + x_0' \cdot x_1') = x_0' \cdot x_1' \]

\[ \exists x_1 \exists x_0 : f = x_0' \cdot x_1' \quad \text{Plug into Eq. 1 to compute } \psi_{Q_1}(q') \]

From BDDs and quantifiers:

\[ (\exists x_1, x_2 : f) \iff (\exists x_1 (\exists x_2 : f)) \]

\[ \exists x : f = f \bigg|_{x = 0} + f \bigg|_{x = 1} \]
Reachability of States - Example

### Transition relation encoding \( \psi_{\delta}(q, q') \)

As a Boolean function
\[
\psi_{\delta}(q, q') = \overline{x_0} \cdot (x_0 \cdot (x_1 + x_1') + x_1 \cdot x_1') + \overline{x_0} \cdot x_0' \cdot \overline{x_1'}
\]

### Compute reachable states:
\[
\psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + (\exists q : \psi_{Q_i}(q) \cdot \psi_{\delta}(q, q'))
\]

#### State encoding
\((x_1, x_0) = \sigma(q)\)

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#### Transition relation encoding

\[
\psi_{Q_1}(q') = \overline{x_1'}
\]

\[
\psi_{Q_2}(q') = \overline{x_1'} + (\exists q : \overline{x_1} \cdot \psi_{\delta}(q, q'))
\]

#### Graph

- \(q_0\) to \(q_1\)
- \(q_0\) to \(q_2\)
- \(q_0\) to \(q_3\)
- \(q_1\) to \(q_2\)
Reachability of States - Example

**State encoding**

\( x_1, x_0 = \sigma(q) \)

<table>
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**Transition relation encoding**

\[ \psi_\delta(q, q') \]

**As a Boolean function**

\[ \psi_\delta(q, q') = x_0' \cdot (x_0 \cdot (x_1 + x_1') + x_1 \cdot x_1') + x_0 \cdot x_0' \cdot x_1' \]

**Compute reachable states:**

\[ \psi_{Q_i+1}(q') = \psi_{Q_i}(q') + (\exists q: \psi_{Q_i}(q) \cdot \psi_\delta(q, q')) \]

**Example**

\( \psi_{Q_1}(q') = x_1' \)

\( \psi_{Q_2}(q') = x_1' + (\exists q: x_1' \cdot \psi_\delta(q, q')) \)

\[ \psi_{Q_2}(q') = x_1' + (\exists q: x_1' \cdot x_0' + x_1' \cdot x_0) = x_1' + x_0' \]

\( Q_2 = Q_1 \cup \{q_1, q_2\} = \{q_0, q_1, q_2\} \)
Reachability of States - Example

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State encoding $(x_1, x_0) = \sigma(q)$

Transition relation encoding $\psi_\delta(q, q')$

As a Boolean function

$$\psi_\delta(q, q') = x_0' \cdot (x_0 \cdot (x_1 + x_1') + x_1 \cdot x_1') + x_0 \cdot x_0' \cdot x_1'$$

Compute reachable states:

$$\psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + ( \exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q, q'))$$

$$\psi_{Q_2}(q') = x_1' + x_0'$$

$$\psi_{Q_3}(q') = x_1' + x_0' + ( \exists q : (x_1 + x_0) \cdot \psi_\delta(q, q'))$$

$$= x_1' + x_0' + x_1' + x_0' = x_1' + x_0'$$

$Q_3 = Q_2 \cup \{q_1, q_2\} = \{q_0, q_1, q_2\}$
It’s always a reachability problem

Or rather

The goal is to transform the problem at hand to encode it as a reachability problem.

Because these can be solved very efficiently

1. Work with sets of states
2. Use characteristic functions to represent sets of states
3. Use ROBDDs to encode characteristic functions
It’s always a reachability problem

Or rather

The goal is to transform the problem at hand to encode it as a reachability problem.

Because these can be solved very efficiently

1. Work with sets of states
2. Use characteristic functions to represent sets of states
3. Use ROBDDs to encode characteristic functions

Comparison of finite automata

1. Compute the set of jointly reachable states
2. Compare the output values of two finite automata
3. …
Your turn to practice!

after the break

1. Familiarise yourself with the equivalence “set of states” \(\equiv\) “characteristic functions”

2. Express system properties using characteristic functions

3. Draw and simplify BDDs to compare a specification and an implementation
Efficient state representation

- Set of states as Boolean function
- Binary Decision Diagram representation

Computing reachability

- Leverage efficient state representation
- Explore successor sets of states

Proving properties

- Temporal logic (CTL)
- Encoding as reachability problem

Next week