Automata & languages

A primer on the Theory of Computation

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The class of regular languages is closed under the

- union
- concatenation
- star

regular operations
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- union
- concatenation
- star

$L_1 \cup L_2$
$L_1 \cdot L_2$
$L_1^*$
Last week, we started to learn about closure and equivalence of regular languages.

\[
\text{REX} \equiv \text{DFA} \approx \text{NFA}
\]
We’ll finish that today then start asking ourselves whether all languages are regular

$L_1 \{0^n1^n | n \geq 0\}$

$L_2 \{w | w \text{ has an equal number of 0s and 1s}\}$

$L_3 \{w | w \text{ has an equal number of occurrences of 01 and 10}\}$

(only one of them actually is)
Advanced Automata
Thu Oct 1

1. Equivalence (the end)
   - DFA
   - NFA
   - Regular Expression

2. Non-regular languages

3. Context-free languages
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Part 1  regular language
Part 2  context-free language
Part 3  turing machine
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Part 2
- regular language
- context-free language
- turing machine
Motivation

• Why is a language such as \( \{0^n1^n \mid n \geq 0\} \) not regular?!?

• It’s really simple! All you need to keep track is the number of 0’s...

• In this chapter we first study context-free grammars
  – More powerful than regular languages
  – Recursive structure
  – Developed for human languages
  – Important for engineers (parsers, protocols, etc.)
Example

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- But there is a pattern.
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- A: Generate palindromes **recursively** as follows:
  - Base case: ε, 0 and 1 are palindromes.
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• Notation: $x \rightarrow \epsilon | 0 | 1 | 0x0 | 1x1$.
  – Each pipe ("|") is an or, just as in UNIX regexp’s.
  – In fact, all palindromes can be generated from $\epsilon$ using these rules.
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Q: How would you generate 11011011?
Context Free Grammars (CFG): Definition

• Definition: A context free grammar consists of \((V, \Sigma, R, S)\) with:
  
  – \(V\): a finite set of variables (or symbols, or non-terminals)
  
  – \(\Sigma\): a finite set of terminals (or the alphabet)
  
  – \(R\): a finite set of rules (or productions)
    
    \(\) of the form \(v \rightarrow w\) with \(v \in V\), and \(w \in (\Sigma \cup V)^*\)
    
    (read: “\(v\) yields \(w\)” or “\(v\) produces \(w\)”)
  
  – \(S \in V\): the start symbol.
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Q: What are \((V, \Sigma, R, S)\) for our palindrome example?
Derivations and Language

• Definition: The derivation symbol “⇒” (read “1-step derives” or “1-step produces”) is a relation between strings in $\mathcal{L}(\Sigma \cup \mathcal{V})*$.
  
  We write $x \Rightarrow y$ if $x$ and $y$ can be broken up as $x = svt$ and $y = swt$ with $v \rightarrow w$ being a production in $R$. 

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Definition: The derivation symbol “⇒*”, (read “derives” or “produces” or “yields”) is a relation between strings in \((\Sigma \cup \mathcal{V})^*\). We write \(x \Rightarrow^* y\) if there is a sequence of 1-step productions from \(x\) to \(y\). I.e., there are strings \(x_i\) with \(i\) ranging from 0 to \(n\) such that \(x = x_0\), \(y = x_n\) and \(x_0 \Rightarrow x_1, x_1 \Rightarrow x_2, x_2 \Rightarrow x_3, \ldots, x_{n-1} \Rightarrow x_n\).
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• Definition: Let \(G\) be a context-free grammar. The context-free language (CFL) generated by \(G\) is the set of all terminal strings which are derivable from the start symbol. Symbolically: \(L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}\)
Example: Infix Expressions

- Infix expressions involving \{+, \times, a, b, c, (, )\}
- \(E\) stands for an expression (most general)
- \(F\) stands for factor (a multiplicative part)
- \(T\) stands for term (a product of factors)
- \(V\) stands for a variable: \(a, b, \text{ or } c\)

- Grammar is given by:
  - \(E \rightarrow T \mid E + T\)
  - \(T \rightarrow F \mid T \times F\)
  - \(F \rightarrow V \mid (E)\)
  - \(V \rightarrow a \mid b \mid c\)

- Convention: Start variable is the first one in grammar \(E\)
Example: Infix Expressions

- Consider the string $u$ given by $a \times b + (c + (a + c))$
- This is a valid infix expression. Can be generated from $E$.

1. A sum of two expressions, so first production must be $E \Rightarrow E + T$
2. Sub-expression $a \times b$ is a product, so a term so generated by sequence $E + T \Rightarrow T + T \Rightarrow T \times F + T \Rightarrow * \ a \times b + T$
3. Second sub-expression is a factor only because a parenthesized sum. $a \times b + T \Rightarrow a \times b + F \Rightarrow a \times b + (E) \Rightarrow a \times b + (E + T) \ldots$
4. $E \Rightarrow E + T \Rightarrow T + T \Rightarrow T \times F + T \Rightarrow F \times F + T \Rightarrow V \times F + T \Rightarrow a \times F + T \Rightarrow a \times V + T \Rightarrow a \times b + T \Rightarrow a \times b + F \Rightarrow a \times b + (E) \Rightarrow a \times b + (E + T) \Rightarrow a \times b + (T + T) \Rightarrow a \times b + (F + T) \Rightarrow a \times b + (V + T) \Rightarrow a \times b + (c + T) \Rightarrow a \times b + (c + F) \Rightarrow a \times b + (c + (E)) \Rightarrow a \times b + (c + (E + T)) \Rightarrow a \times b + (c + (T + T)) \Rightarrow a \times b + (c + (F + T)) \Rightarrow a \times b + (c + (a + T)) \Rightarrow a \times b + (c + (a + F)) \Rightarrow a \times b + (c + (a + V)) \Rightarrow a \times b + (c + (a + c))$
Left- and Right-most derivation

- The derivation on the previous slide was a so-called **left-most derivation**.

- In a **right-most derivation**, the variable most to the right is replaced.
  
  \[ E \Rightarrow E + T \Rightarrow E + F \Rightarrow E + (E) \Rightarrow E + (E + T) \Rightarrow \text{etc.} \]
Ambiguity

- There can be a lot of ambiguity involved in how a string is derived.
- Another way to describe a derivation in a unique way is using derivation trees.
Derivation Trees

- In a derivation tree (or parse tree) each node is a symbol. Each parent is a variable whose children spell out the production from left to right. For example $v \rightarrow abcdefg$:

- The root is the start variable.
- The leaves spell out the derived string from left to right.
Derivation Trees

- On the right, we see a derivation tree for our string $a \times b + (c + (a + c))$

- Derivation trees help understanding semantics! You can tell how expression should be evaluated from the tree.
Ambiguity

\[
\begin{align*}
\text{<sentence>} & \rightarrow \quad \text{<action>} \mid \text{<action>} \text{ with } \text{<subject>} \\
\text{<action>} & \rightarrow \quad \text{<subject>}\text{<activity>} \\
\text{<subject>} & \rightarrow \quad \text{<noun>} \mid \text{<noun>} \text{ and } \text{<subject>} \\
\text{<activity>} & \rightarrow \quad \text{<verb>} \mid \text{<verb>}\text{<object>} \\
\text{<noun>} & \rightarrow \quad \text{Hannibal} \mid \text{Clarice} \mid \text{rice} \mid \text{onions} \\
\text{<verb>} & \rightarrow \quad \text{ate} \mid \text{played} \\
\text{<prep>} & \rightarrow \quad \text{with} \mid \text{and} \mid \text{or} \\
\text{<object>} & \rightarrow \quad \text{<noun>} \mid \text{<noun>}\text{<prep>}\text{<object>} \\
\end{align*}
\]

- Clarice played with Hannibal
- Clarice ate rice with onions
- Hannibal ate rice with Clarice

Q: Are there any suspect sentences?
Ambiguity

• A: Consider “Hannibal ate rice with Clarice”

• This could either mean
  – Hannibal and Clarice ate rice *together*.
  – Hannibal ate rice and *ate* Clarice.

• This ambiguity arises from the fact that the sentence has two different parse-trees, and therefore two different interpretations:
Hannibal and Clarice Ate
Hannibal the Cannibal

sentence

action

subject

noun

H a n n i b a l

Hannibal

activity

verb

te

ate

deast

object

noun

rice

Clarice

c

with

pre

prep

noun

object
Ambiguity: Definition

• Definition:

A string $x$ is said to be ambiguous relative to the grammar $G$ if there are two essentially different ways to derive $x$ in $G$.
  – $x$ admits two (or more) different parse-trees
  – equivalently, $x$ admits different left-most [resp. right-most] derivations.

• A grammar $G$ is said to be ambiguous if there is some string $x$ in $L(G)$ which is ambiguous.
Ambiguity: Definition

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A string $x$ is said to be ambiguous relative the grammar $G$ if there are two essentially different ways to derive $x$ in $G$.
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• A grammar $G$ is said to be ambiguous if there is some string $x$ in $L(G)$ which is ambiguous.

• Question: Is the grammar $S \rightarrow ab \mid ba \mid aSb \mid bSa \mid SS$ ambiguous?
  – What language is generated?
Ambiguity

- Answer: $L(G) =$ the language with equal no. of $a'$ s and $b'$ s
- Yes, the language is ambiguous:
CFG’s: Proving Correctness

• The recursive nature of CFG’s means that they are especially amenable to correctness proofs.

• For example let’s consider the grammar

\[ G = (S \rightarrow \varepsilon \mid ab \mid ba \mid aSb \mid bSa \mid SS) \]

• We claim that \( L(G) = L = \{ x \in \{a,b\}^* \mid n_a(x) = n_b(x) \} \), where \( n_a(x) \) is the number of \( a \)'s in \( x \), and \( n_b(x) \) is the number of \( b \)'s.

• \textbf{Proof}: To prove that \( L = L(G) \) is to show both inclusions:
  
  \( i. \quad L \subseteq L(G) \): Every string in \( L \) can be generated by \( G \).
  
  \( ii. \quad L \supseteq L(G) \): \( G \) only generate strings of \( L \).
    
    - This part is easy, so we concentrate on part i.
Proving $L \subseteq L(G)$

- $L \subseteq L(G)$: Show that every string $x$ with the same number of $a$’s as $b$’s is generated by $G$. Prove by induction on the length $n = |x|$.
- Base case: The empty string is derived by $S \rightarrow \varepsilon$.
- Inductive hypothesis: Assume $n > 0$. Let $u$ be the smallest non-empty prefix of $x$ which is also in $L$.
  - Either there is such a prefix with $|u| < |x|$, then $x = uv$ whereas $v \in L$ as well, and we can use $S \rightarrow SS$ and repeat the argument.
  - Or $x = u$. In this case notice that $u$ can’t start and end in the same letter. If it started and ended with $a$ then write $x = ava$. This means that $v$ must have 2 more $b$’s than $a$’s. So somewhere in $v$ the $b$’s of $x$ catch up to the $a$’s which means that there’s a smaller prefix in $L$, contradicting the definition of $u$ as the smallest prefix in $L$. Thus for some string $v$ in $L$ we have $x = avb$ OR $x = bva$. We can use either $S \rightarrow aSb$ OR $S \rightarrow bSa$. 

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Designing Context-Free Grammars

• As for regular languages this is a creative process.

• However, if the grammar is the union of simpler grammars, you can design the simpler grammars (with starting symbols $S_1$, $S_2$, respectively) first, and then add a new starting symbol/production $S \rightarrow S_1 \mid S_2$.

• If the CFG happens to be regular as well, you can first design the FA, introduce a variable/production for each state of the FA, and then add a rule $x \rightarrow ay$ to the CFG if $\delta(x,a) = y$ is in the FA. If a state $x$ is accepting in FA then add $x \rightarrow \varepsilon$ to CFG. The start symbol of the CFG is of course equivalent to the start state in the FA.

• There are quite a few other tricks. Try yourself…