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Computer Systems

— Solution to Assignment 12 —

1 Game Theory

Quiz

1.1 Selling a Franc

We assume that there are two bidders, b_1 and b_2. If b_1 bids 5 rappen, his gain is 95 rappen. Now b_2 is inclined to bid 10 rappen and gains 90 rappen. This can continue until b_1 bids 95 rappen. Bidder b_2 now has the choice of losing 90 rappen (her last bid) or coming out even. Since she is a rational player, she will bid 1 franc. Bidder b_1 now faces a similar choice. Either he loses 95 rappen or he bids and has a chance of only losing 5 rappen. Since he is a rational player, he will bid 1.05 franc. This bidding war will continue indefinitely (or until one bidder runs out of money).

There are a few ways the bidders could have avoided this situation. Apart from the obvious, simply do not play, they could have also colluded. One bidder bids 5 rappen for the franc and the bidders will simply split the money they made. This requires that the bidders can trust each other. As you can guess, there are games that anticipate collaboration.
There exists however a strategy, which is profitable even for a non-colluding bidder. If the first bidder bids 95 rappen, she will win 5 rappen, because nobody else will also bid. Why will nobody else bid? If another bidder bids more, he will certainly not bid more than 1 franc, because this will yield a negative payoff. Instead, he could bid 1 franc. But then, the first bidder will bid 1.05 francs to minimize her loss. And then the game continues as outlined at the beginning and both bidders will incur a loss. Therefore, no rational player will bid 1 franc in this scenario.

Basic

1.2 Selfish Caching

To be sure that we find every Nash Equilibrium, we explicitly write down every best response.

i. The best response strategies are

u: cache only if nobody else does. (B1)

v: cache if neither u nor x cache. (B2)

w: cache unless u caches. (B3)

x: cache if neither u nor v cache. (B4)

Nash equilibrium. If we assume that u plays $Y_u = 1$ (u caches) the system can only be in a NE if $Y_v = Y_w = Y_x = 0$ due to (B1). Since for all v, w, and x it is the best response not to cache if u does, $x = (1000)$ is an Nash equilibrium. If $Y_u = 0$ then (B3) implies $Y_w = 1$. If furthermore, $Y_v = 1$ it must hold that $Y_x = 0$ due to (B2). This does not conflict with (B4), and (0110) constitutes another NE. Last, if $Y_v = 0$ then (B2) implies $Y_x = 1$, which is also okay with (B4). Hence (0011) is also a NE.

$$NE = \{(1000), (0110), (0011)\}$$

Price of anarchy. The social optimum is achieved in strategy profile (1000), namely $OPT = cost(1000) = 1 + \frac{1}{2} + \frac{3}{8} + \frac{3}{4} = \frac{21}{8}$. Since (1000) is also a Nash equilibrium we immediately get that $OPoA = 1$. The worst-case price of anarchy is

$$PoA = \frac{cost(0110)}{OPT} = \frac{\frac{1}{2} + 1 + 1 + \frac{7}{8}}{\frac{21}{8}} = \frac{9}{7} \approx 1.286.$$

ii. The best response strategies are

u: cache only if nobody else does. (B1)

v: cache unless u caches. (B2)

w: cache unless x caches. (B3)

x: cache if neither u nor w cache. (B4)

Nash equilibrium. If we assume that u plays $Y_u = 1$ (u caches) the system can only be in a NE if $Y_v = Y_w = Y_x = 0$ due to (B1). However, $Y_x = 0$ implies that $Y_w = 1$ due to (B3), and hence there can be no NE with $Y_u = 1$. In any NE it must hold that $Y_u = 0$. Consequently, it must hold that $Y_v = 1$ from (B2). Now if $Y_w = 1$ (B3) implies that x does not cache. This does not infringe rule (B4), and thus $x = (0110)$ is a Nash equilibrium. If $Y_w = 0$ then (B4) implies that x caches. As thus, rule (B3) is not violated $x = (0101)$ is also a Nash equilibrium.

Price of anarchy. The social optimum is achieved in strategy profile (0110), namely $OPT = cost(0110) = \frac{1}{3} \cdot 0.2 + 1 + 1 + \frac{1}{2} \cdot 0.2 = 2.1\overline{6}$. Since (0110) is also a Nash equilibrium we get that the optimistic price of anarchy is 1. The worst-case price of anarchy is

$$PoA = \frac{cost(0101)}{OPT} = \frac{1/3 \cdot 0.2 + 1 + 0.2 + 1}{2.1\overline{6}} = \frac{68}{65} \approx 1.046$$

1.3 Selfish Caching with variable caching cost

We define D_i to be the set of nodes that cover node i. A node j *covers* node i if and only if $c_{i \leftarrow j} < \alpha_i$, i.e., node i prefers accessing the object at node j than caching it. Convince yourself that a strategy profile is a Nash Equilibrium if and only if for each node i it holds that

- if $Y_i = 1$ then $Y_j = 0$ for all $j \in D_i$, and
- if $Y_i = 0$ then $\exists j \in D_i$ with $Y_j = 1$.

i. $D_u = \emptyset$, $D_v = \{u, w\}$, $D_w = \{u\}$. D_u being empty implies $Y_u = 1$ (i.e. caches the file). Hence $Y_v = 0$, and $Y_w = 1$. $NE = \{(101)\}$. $PoA = 1$ since (101) is also the social optimum strategy.

ii. $D_u = \{v\}$, $D_v = \{u\}$, $D_w = \{u, v\}$. If $Y_u = 1$, then $Y_v = 0$ and $Y_w = 0$. If $Y_u = 0$, then $Y_v = 1$. Hence $Y_w = 0$. The equilibria are $NE = \{(100), (010)\}$.

$$PoA = \frac{cost(100)}{cost(010)} = \frac{3 + 1 + 8/3}{3/2 + 3/2 + 5/3} = \frac{40}{28} \approx 1.43$$

Dominant strategies. Every dominant strategy profile is also a Nash equilibrium. Hence we only have to check the computed NEs whether they consist of dominant strategies only.

Let us consider game i. Since every dominant strategy profile is also a Nash Equilibrium, it suffices to consider the NE. The game has no dominant strategy profile. Profile (101) is no dominant strategy profile in game i. since, although $Y_u = 1$ is the dominant strategy for u, $Y_v = 0$, and $Y_w = 1$ are not dominant strategies for v and w. If $Y_v = 1$, then it would be the best response of w to set $Y_w = 0$. Game ii: Since the decision of node u whether to cache depends on the decision of node v, this is not a dominant strategy. Therefore, this game has no dominant strategy profile.

Advanced

1.4 Matching Pennies

The bi-matrix of the game with Tobias as row player, and Stephan as column player looks as follows:

	H	T
H	1 , -1	-1 , 1
T	-1 , 1	1 , -1

This zero-sum game has no pure Nash equilibrium. For the mixed NEs, Tobias plays heads (H) with probability p, tails (T) with probability $1-p$. Stephan plays H with probability q, and T with probability $1-q$. We get the expected utility functions Γ:

$$\begin{aligned}\Gamma_T(p,q) &= p(q-(1-q)) + (1-p)(-q+(1-q)) = (4q-2)\cdot p + 1 - 2q \\ \Gamma_S(p,q) &= q(-p+(1-p)) + (1-q)(p-(1-p)) = (2-4p)\cdot q + 2p - 1\end{aligned}$$

If Stephan plays $q = 1/2$ the term $4q-2$ equals 0, and any choice of p will yield the same payoff for Tobias. If Tobias plays $p = 1/2$ then any choice of q is a best response for Stephan. Thus $(p,q) = (1/2, 1/2)$ is a mixed NE. Note that for any choice of $p > 1/2$, Stephan's best response is to choose $q = 0$. For a $p < 1/2$ Stephan would choose $q = 1$. However, Tobias' best response to $q > 1/2$ is $p = 1$, and $p = 0$ if $q < 1/2$. Hence $(p,q) = (1/2, 1/2)$ is the only pair of mutual best responses.

1.5 PoA Classes

Let I^n be an instance of $\mathcal{A}^n_{[a,b]}$ that maximizes the price of anarchy, i.e. $PoA(\mathcal{A}^n_{[a,b]}) = PoA(I^n)$. Let $x, y \in X$ be two strategy profiles in I^n such that $PoA(I^n) = cost(y)/cost(x)$. We show the claim by constructing an instance $\hat{I}^n \in \mathcal{W}^n_{[\frac{1}{b},\frac{1}{a}]}$ out of I^n for which it holds that $PoA(\hat{I}^n) \geq \frac{a}{b} PoA(I^n) = \frac{a}{b} PoA(\mathcal{A}^n_{[a,b]})$. We construct $\hat{I}^n$ by setting $d_i = 1/\alpha_i$, $\hat{\alpha}_i = 1$ where α_i are the placement costs (for local caching) of player i in I^n. All edges remain as in I^n. This game has the same Nash equilibria as I^n since the cover sets D_i (nodes for which we do not cache if these cache already) for each peer stay the same. A peer j is in D_i iff $c_{i \leftarrow j} < \alpha_i$, or $c_{i \leftarrow j}/\alpha_i < 1$ respectively. We get the bound by comparing the performance of the two strategies x, y that produce the PoA in I^n in $\hat{I}^n$. Note that x is not necessarily a social optimum in $\hat{I}^n$, but y is a Nash equilibrium also in $\hat{I}^n$, because the cover sets are the same.

$$
\begin{aligned}
PoA(\hat{I}^n) &\geq \frac{\hat{cost}(y)}{\hat{cost}(x)} = \frac{\sum_{i=1}^{n}\left(y_i + (1-y_i)\frac{c_i(y)}{\alpha_i}\right)}{\sum_{i=1}^{n}\left(x_i + (1-x_i)\frac{c_i(x)}{\alpha_i}\right)} && (1)\\
&= \frac{b \cdot a \sum_{i=1}^{n}\left(y_i + (1-y_i)\frac{c_i(y)}{\alpha_i}\right)}{b \cdot a \sum_{i=1}^{n}\left(x_i + (1-x_i)\frac{c_i(x)}{\alpha_i}\right)} && (2)\\
&\geq \frac{a \sum_{i=1}^{n}\left(y_i\alpha_i + (1-y_i)c_i(y)\right)}{b \sum_{i=1}^{n}\left(x_i\alpha_i + (1-x_i)c_i(x)\right)} && (3)\\
&= \frac{a \cdot cost(y)}{b \cdot cost(x)} = \frac{a}{b} PoA(I^n) && (4)
\end{aligned}
$$

$\hat{cost}(x)$ denotes the cost function in $\hat{I}^n$. x_i, and y_i are either 1 or 0. x_i equals 1 if player i caches in strategy profile x, and 0 if she does not. With $c_i(y)$ we denote the cost of node i if it access the file remotely in strategy y. For step (3) we exploit the fact that $b \geq \alpha_i$ and $a \leq \alpha_i$ for all i.

2 Quorum Systems

2.1 The Resilience of a Quorum System

a) No such quorum system exists. According to the definition of a quorum system, every two quorums of a quorum system intersect, so at least one server is part of both quorums. The fact that all servers of a particular quorum fail implies that in each other quorum at least one server fails, namely the one which lies in the intersection. Therefore, it is not possible to achieve a quorum anymore and the quorum system does not work anymore.

b) Just 1—as soon as 2 servers fail, no quorum survives.

c) Imagine a quorum system in which all quorums overlap exactly in one single node; i.e. each element of the powerset of the remaining $n-1$ nodes joined with this special node is a quorum. This gives 2^{n-1} quorums.
Can there be more? No! Consider a set from the powerset of n servers. Its complement cannot be a quorum as well, as they do not overlap. So, from each such couple, at most one set can be part of the quorum system. This gives an upper bound of $2^n/2 = 2^{n-1}$ quorums.

2.2 A Quorum System

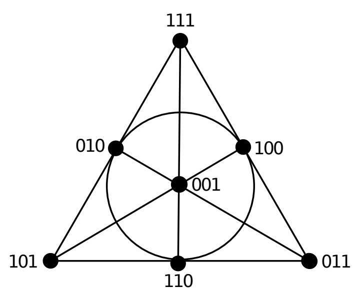

Figure 1: Quorum System

a) This quorum system consists of 7 quorums. As work is defined as the minimum expected number of servers in an accessed quorum (over all access strategies), this system's work is

3 (all strategies induce the same work on a system where all quorums are the same size). Observe that all nodes are in precisely 3 quorums, so the uniform access strategy induces the same load on all nodes. Since the quorum system is also 3-uniform, by exercise 3 it follows that the uniform strategy is optimal; it's load being 3/7.

b) The resilience is $R(\mathcal{S}) = 2$. Proof: every node is in exactly 3 quorums, so 2 nodes can be contained in at most $2 \cdot 3 = 6 < 7 = |\mathcal{S}|$ quorums, thus, if no more than 2 nodes fail, there will be at least 1 quorum without a faulty node. If, on the other hand, for example, the nodes 101, 010 and 111 fail, no quorum can be achieved; see also exercise 1a).

2.3 S-Uniform Quorum Systems

Definitions:

s-uniform: A quorum system $\mathcal{S}$ is *s-uniform* if every quorum in $\mathcal{S}$ has exactly s elements.
Balanced access strategy: An access strategy Z for a quorum system $\mathcal{S}$ is *balanced* if it satisfies $L_Z(v_i) = L$ for all $v_i \in V$, for some value L.

Claim: An s-uniform quorum system $\mathcal{S}$ reaches an optimal load with a balanced access strategy, if such a strategy exists.

a) In an s-uniform quorum system each quorum has exactly s elements, so independently of which quorum is accessed, s servers have to work. Summed up over all servers we reach a total load of s, which is the work of the quorum system. As the load induced by an access strategy is defined as the maximum load on any server, the best strategy would be to evenly distribute this work on all servers. If such a strategy exists, then it is therefore optimal.

b) Let $V = \{v_1, v_2, ..., v_n\}$ be the set of servers and $\mathcal{S} = \{Q_1, Q_2, ..., Q_m\}$ an s-uniform quorum system on V. Let Z be an access strategy, thus it holds that: $\sum_{Q \in \mathcal{S}} P_Z(Q) = 1$. Furthermore, let $L_Z(v_i) = \sum_{Q \in \mathcal{S}; v_i \in Q} P_Z(Q)$ be the load of server v_i induced by Z.

Then it holds that:

$$\sum_{v_i \in V} L_Z(v_i) = \sum_{v_i \in V} \sum_{Q \in \mathcal{S}; v_i \in Q} P_Z(Q) = \sum_{Q \in \mathcal{S}} \sum_{v_i \in Q} P_Z(Q)$$
$$= \sum_{Q \in \mathcal{S}} P_Z(Q) \cdot |Q| \overset{*}{=} \sum_{Q \in \mathcal{S}} P_Z(Q) \cdot s = s \cdot \sum_{Q \in \mathcal{S}} P_Z(Q) = s$$

The transformation marked with an asterisk uses the uniformity of the quorum system.

To minimize the maximal load on any server, the optimal strategy would be to evenly distribute this load on all servers. Thus, if a balanced access strategy exists, this leads to an optimal system load of s/n.

Note: A balanced access strategy does not always exist for example for the following 2-uniform quorum system: $V = \{1, 2, 3\}$, $\mathcal{S} = \{\{1, 2\}, \{1, 3\}\}$. We have $\min\{L_Z(2), L_Z(3)\} < L_Z(1) = 1$ for any access strategy on this system.