1 Pancake Networks

In the lecture, you have encountered several different graphs as underlying network structures for peer-to-peer (P2P) networks. Here, we will look at another prominent example, the Pancake graph $P_n$.

The pancake graph $P_n$ is defined as follows: The vertex set is

$$V(P_n) = \{(v_1, v_2, \ldots, v_n) \mid v_i \in \{1, \ldots, n\} \text{ and } v_i \neq v_j \forall i \neq j\}.$$ 

In other words, $V(P_n) = S_n$, the group of all permutations on $n$ elements. There exists an edge of dimension $i$ for $2 \leq i \leq n$ when

$$e_i = \{(v_1, \ldots, v_i, \ldots, v_n), (w_1, \ldots, w_i, \ldots, w_n)\} \in E(P_n) \iff w_j = v_{i-j+1} \text{ for } 1 \leq j \leq i \text{ and } w_j = v_j \text{ for } i < j \leq n,$$

or, we can say that an edge $e_i$ represents a prefix reversal

$$(v_1, \ldots, v_i, v_{i+1}, \ldots, v_n) \longleftrightarrow (v_1, \ldots, v_i, v_{i+1}, \ldots, v_n). \quad (1)$$

For the following questions, where appropriate, give your answers in terms of $N := |V(P_n)|$ (approximately), the number of vertices, as well as $n$.

a) Draw (nicely!) $P_n$ for $n = 2, 3, 4$. Try to describe a pattern for drawing $P_n$ for any $n$.

b) What is the degree of each vertex in $P_n$?

c) Can you give bounds on the diameter $D(P_n)$ of the pancake network?

d) Show that $P_n$ is Hamiltonian for $n \geq 3$.

e) How can the pancake graph be used to implement a distributed hash table (DHT)? In other words, where are files, indexed by bitstrings of a certain length $b$, stored in the pancake graph, and how can these files be looked up (given the corresponding bitstring)?

The pancake graph has been proposed for P2P networks partly because of the properties analyzed in this exercise.

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1 A well-known paper about pancake graphs was originally written in 1976 by a 21 year old college dropout (later to become a famous entrepreneur) and a Ph.D. student (later to become a famous scholar):

2 You can ignore churn in this exercise.