For this exercise, you may use the following Chernoff bound.¹

**Theorem 1 (Chernoff’s Bound)**

Let \( X := \sum_{i=1}^{N} X_i \) be the sum of \( N \) independent \( 0 - 1 \) random variables \( X_i \). Then with high probability \(^2\) (w.h.p.)

\[
|X - \mathbb{E}[X]| \in O\left(\log n + \sqrt{\mathbb{E}[X] \log n}\right).
\]

## 1 Diameter of the Augmented Grid

Recall the network from the lecture where nodes were arranged in a grid and each node had an additional directed link to an uniformly and independently at random drawn node in the network (i.e., \( \alpha = 0 \)). In the lecture, a proof of the fact that such a network has diameter \( O(\log n) \) w.h.p. was sketched. We will now fill in the details.

### a) Show that \( O(cn/\log n) \) many nodes are enough to guarantee with high probability that at least one of their random links connects to a given set of \( \Omega(\log^2 n) \) nodes. Prove this (i) by direct calculation and (ii) using Chernoff’s bound.

**Hint:** Use that \( 1 - p \leq e^{-p} \) for any \( p \).

### b) Suppose for some node set \( S \) we have that \( |S| \in \Omega((\log n)^2) \cap o(n) \) and denote by \( H \) the set of nodes hit by their random links. Prove that \( H \) and together with its grid neighbors contains w.h.p. \( (5 - o(1))|S| \) nodes!

**Hint:** Observe that independently of all previous random choices, each new link has at least a certain probability \( p \) of connecting to a node whose complete neighborhood has not been reached yet. Then use Chernoff’s bound on the sum of \( |S| \) many variables.

### c) Infer from b) that starting from \( \Omega(\log^2 n) \) nodes, with each hop the number of reached nodes w.h.p. more than doubles, as long as we have still \( O(n/\log n) \) nodes (regardless of the constants in the \( O \)-notation).

**Hint:** Play with the constant \( c \) in the definition of w.h.p. and use the union bound.

### d) Conclude that the diameter of the network is w.h.p. in \( O(\log n) \).

¹Chernoff-type and similar probability bounds are very powerful tools that allowed to design a plethora of randomized algorithms that almost guarantee success. Frequently this “almost” makes a huge difference in e.g. running time and/or approximation quality.

²I.e., with probability at least \( 1 - 1/n^c \) for any fixed constant \( c > 0 \). The “fixed” here indicates that we typically see \( c \) as a constant with regard to \( O \)-notation. Thus, if someone names a constant \( c \), then for instance the statement “the algorithm requires \( O(\log n) \) rounds with high probability” translates to “the algorithm requires \( O(\log n) \) rounds with probability at least \( 1 - 1/n^{e^2} \),” where the \( O(\log n) \) could mean e.g. \( 5e^2 \log n + e^{e^2} + 1001 \).
2 Greedy Routing in the Augmented Grid

Consider now \( \alpha = 2 \), i.e., the random link of node \( u \) connects it to node \( w \) with probability \( d(u, w)^{-2}/\sum_{v \in \mathcal{V}\setminus\{u\}} d(u, v)^{-2} \). In the lecture, we saw that with probability \( \Omega(1/\log n) \), in each step we get to the next phase when we employ greedy routing. Hence, the expected number of steps is in \( O(\log^2 n) \). Prove that the same bound on the number of steps holds w.h.p.!