Cheap Labor Can Be Expensive

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The Problem

“Nash Equilibrium”
The Problem
The Problem

Total price: 5
The Problem

Total price: 10
Markets

- Set of agents “E”
- Each agent $e \in E$ has a cost $c(e)$ and bid $b(e)$
- Customer wants to hire a team of agents
- **Feasible sets** “F” are teams of agents capable of getting the job done
Feasible Sets
Cheap Labor Cost

Total price: $0

→ Cheap Labor Cost of this market is $\frac{10}{5} = 2$
Cheap Labor Cost

• Cheap labor cost for a market $M$:

$$\frac{p_M}{\min_{M' \subseteq M} p_{M'}}$$

• $p_M$ := total price of $M$
Questions up to this point?
GreedyAlg

1. Find the cheapest feasible set $S \in F$ with respect to costs
GreedyAlg

2. For each $e \in E$, initialize $b(e)$ to $c(e)$
GreedyAlg

3. For each $e \in S$:
   - Raise $b(e)$ until there is $S' \in F$ such that $e \notin S'$ and $b(S) = b(S')$
GreedyAlg

1. Find the cheapest feasible set $S \in F$ with respect to costs
2. For each $e \in E$, initialize $b(e)$ to $c(e)$
3. For each $e \in S$:
   - Raise $b(e)$ until there is $S' \in F$ such that $e \not\in S'$ and $b(S) = b(S')$
4. Output the bids $b$ and the winning set $S$
Tight sets

- For any NE $b$ with winning set $S$:
  - For any $e \in S$, there is another winning feasible set $S' \in F$ with $e \in S'$ and $b(S) = b(S')$
  - These feasible sets are called **tight sets**.
Upper Bound

• The cheap labor cost of any market is at most $|S|$, where $S \in F$ is a feasible set with minimum total cost

Here: $|S| = 3$
Proof of Upper Bound

• It suffices to show:
  – For any market $M$, NE $b$ with winning set $S$, for any submarket $M'$, best NE $b'$ with winning set $S'$

\[
b(S) \leq |S| \cdot b'(S')
\]

\[
\frac{b(S)}{b'(S')} \leq |S|
\]

(we choose $b$ and $S$ to be computed by GreedyAlg)
Proof of Upper Bound

Case 1: $e \in S' \setminus S$

- $b(e) = c(e)$
- $b(S' \setminus S) = c(S' \setminus S)$
- $b(S \setminus S') \leq b(S' \setminus S)$
- $c(S' \setminus S) \leq b'(S' \setminus S)$
- $b(S \setminus S') \leq b'(S' \setminus S)$
Proof of Upper Bound

Case 2: $e \in S' \cap S$

- For each such $e$ there exists a tight set $S''$ (in $F'$) such that $e \not\in S''$ and $b'(S') = b'(S'')$.
- We claim $b(e) \leq b'(S')$. Otherwise:
  \[
  b(S) = b(S \setminus S'') + b(S \cap S'')
  > b'(S') + b(S \cap S'') \quad \text{[reverse claim]}
  = b'(S'') + b(S \cap S'') \quad \text{[bid behavior]}
  \geq c(S'') + b(S \cap S'') \quad \text{[GreedyAlg]}
  \geq c(S'' \setminus S) + b(S \cap S'')
  = b(S'' \setminus S) + b(S \cap S'')
  = b(S'') \quad \text{[contradiction: S is the winning set]}
Proof of Upper Bound

Case 1 (e ∈ S’ \ S):  b(S \ S’) ≤ b’(S’ \ S)

Case 2 (e ∈ S’ ∩ S):  b(e) ≤ b’(S’)

Putting the cases together:

\[ b(S) = b(S \setminus S’) + b(S \cap S’) \]
\[ \leq b’(S’ \setminus S) + |S \cap S’| \cdot b’(S’) \]
\[ \leq |S| \cdot b’(S’) \]
Perfect Bipartite Matching Markets

- Customer wants to buy edges to obtain a perfect matching in a bipartite graph.
Perfect Bipartite Matching Markets

\[ p_M = k \quad p_{M'} = 1 \quad \text{Cheap labor cost} = k = O(|V|) \]
Perfect Bipartite Matching Markets

\[ u_0 \rightarrow v_0 \quad u_i \rightarrow v_i \quad u_{i'} \rightarrow v_{i'} \]

\[ 1 \]
Matroid Markets

- Agents and feasible sets form a matroid \((E, F)\) \((F \subseteq P(E)\) with a bunch of special rules\)
- Cheap labor cost is always 1.
- Natural Occurrence: buying spanning trees
Path Markets

• Purchasing an s-t path in a directed graph
Path Markets

- Observation: There are always at least 2 edge-disjoint paths $P_1$ and $P_2$ with $b(P_1) = b(P_2) = b(P)$, where $P$ is the winning path.
Path Markets

• Proof idea:
  – There always are “tight paths” (tight sets)
    For any $e \in S$, there is another winning feasible set $S' \in F$ with $e \in S'$ and $b(S) = b(S')$
  – Any prefix of a tight path is optimal (otherwise the winning path would not be winning).
  – The union of all tight paths only contains optimal s-t paths and is two-connected.
Path Markets

• Proposition:
  – Pick the two cheapest paths by cost, $P_1$ and $P_2$.
  – $\min_{G' \subseteq G'} p_{G'} = \max\{c(P_1), c(P_2)\}$

($p_G :=$ total price of $G$)
Path Markets

• Now observe that for the two cheapest paths by cost, P1 and P2, \( c(P_1) + c(P_2) \) gives an upper bound for \( p_G \).

• Thus, \( p_G \leq c(P_1) + c(P_2) \leq 2 \cdot \max\{c(P_1), c(P_2)\} = 2 \cdot p_{G*} \)

\[ \Rightarrow \] The cheap labor cost for path markets is at most 2.
Path Markets

• This bound is tight:
Conclusion

• Short paper stuffed with proofs
• Exhaustive study of “cheap labor cost” for non-cooperative markets
  – General upper bound $|S|$
  – Values for common market types
Thanks for your attention!

Questions?