1 Communication Complexity of Set Disjointness

a) We obtain

\[ M_{\text{DISJ}} = \begin{bmatrix}
000 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
001 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
010 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
011 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
100 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
101 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
110 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
111 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\text{DISJ} \quad 000 \quad 001 \quad 010 \quad 011 \quad 100 \quad 101 \quad 110 \quad 111 \quad \leftarrow x \]

b) When \( k = 3 \), a fooling set of size 4 for DISJ is e.g.

\[ S_1 := \{(000, 111), (001, 110), (010, 101), (100, 001)\}. \]

Entries in \( M_{\text{DISJ}} \) corresponding to elements of \( S_1 \) are marked dark gray. However, a fooling set does not always need to be on a diagonal of the matrix. An example for such a set is

\[ S_2 := \{(001, 110), (010, 001), (011, 100), (100, 010)\}, \]

and marked light gray in \( M_{\text{DISJ}} \).

c) In general, \( S := \{(x, \overline{x}) \mid x \in \{0, 1\}^k\} \) is a fooling set for DISJ. To prove this, we note: If \( y > \overline{x} \) then there is always an index \( i \) such that \( x_i = y_i = 1 \) and we conclude \( \text{DISJ}(x, y) = 0 \). Second, we note for any elements \( (x_1, y_1), (x_2, y_2) \) of any fooling set that \( x_1 \neq x_2 \). Otherwise it was \( (x_1, y_j) = (x_2, y_j) \) for \( j \in \{1, 2\} \) and thus \( f(x_2, y_1) = f(x_1, y_2) = f(x_1, y_1) = f(x_2, y_2) =: z \) which contradicts the definition of a fooling set. Similarly it is \( y_1 \neq y_2 \).

- For any \( (x, y) \in S \) it is \( \text{DISJ}(x, y) = 1 \).
- Now consider any \( (x_1, y_1) \neq (x_2, y_2) \in S \). Since \( x_1 \neq x_2 \) and \( y_1 \neq y_2 \), we conclude that either \( y_2 > \overline{x_1} \), in which case \( \text{DISJ}(x_1, y_2) = 0 \), or \( y_1 > \overline{x_2} \) causing \( \text{DISJ}(x_2, y_1) = 0 \).
2 Distinguishing Diameter 2 from 4

a) • Choosing \( v \in L \) takes \( O(D) \): Use any leader election protocol from the lecture. E.g. the node with smallest ID in \( L \) can be elected as a leader. Then this node will be \( v \).
• Computing a BFS tree from a vertex usually takes \( O(D) \). Since in our setting all graphs are guaranteed to have constant diameter, the time required for this is \( O(1) \). As node \( v \) is in \( L \), at most \( |N_1(v)| \leq s \) executions of BFS are performed. These can be started one after each other and yield a complexity of \( O(s) \).
• The comment states: Computing an \( H \)-dominating set \( DOM \) takes time \( O(D) = O(1) \).
• Since \( |DOM| \leq \frac{n \log n}{s} \), the time complexity of computing all BFS trees from each vertex in \( DOM \) (one after each other) is \( O\left(\frac{n \log n}{s}\right) \).
• Checking whether all trees have depth of at most 2 can be done in \( O(D) = O(1) \) as well: Each node knows its depth in any of the computed trees. If its depth is 3 or 4, it floods “diameter is 4” to the graph. If a node gets such a message from several neighbors, it only forwards it to those from which it did not receive it yet. If any node did not receive message “diameter is 4” after 4 rounds, it decides that the diameter is 2. Otherwise it decides that the diameter is 4. This decision will be consistent among all nodes.
• By adding all these runtimes, we conclude that the total time complexity of Algorithm 2-vs-4 is \( O\left(s + \frac{n \log n}{s}\right) \).

b) By deriving \( O\left(s + \frac{n \log n}{s}\right) \) as a function of \( s \) we can argue that \( O\left(s + \frac{n \log n}{s}\right) \) is minimal for \( s = \sqrt{n \log n} \). Thus the runtime of the Algorithm is \( O(\sqrt{n \log n}) \).

c) Since in this case no BFS tree can have depth larger than 2 the algorithm returns “diameter is 2”.

d) Using the triangle inequality we obtain that \( d(w, v) \geq d(u, v) - d(w, v) = 3 \) thus the BFS tree of \( w \) has at least depth 3. Therefore Algorithm 2-vs-4 decides “diameter is 4”.

e) Let \( w \) be the leader elected in step 2 of Algorithm 2-vs-4. If the BFS started in \( w \) has depth at least 3, we are done. In the other case it is \( d(u, w) \leq 2 \). Using d) we conclude that \( d(u, w) = 2 \). Let \( w' \) be a node that connects \( u \) to \( w \). Since \( w' \in N_1(w) \), Algorithm 2-vs-4 executes a BFS from \( w' \). Then we apply d) using that \( w' \in N_1(u) \).

f) Since \( DOM \) is a dominating set for \( H = V \setminus L = V \), it follows immediately that the algorithm executes a BFS from a node \( w \in DOM \cap N_1(u) \neq \emptyset \). Now apply d).

g) A careful look into the construction of family \( G \) reveals that we essentially showed an \( \Omega\left(n/\log n\right) \) lower bound to distinguish diameter 2 from 3. Since the graphs considered here cannot have diameter 3, the studied algorithm does not contradict this lower bound.

h) Consider a clique (with \( n \) nodes, \( n \) large enough) and remove an arbitrary edge \( (u, v) \). Since \( d(u, v) = 2 \), the graph has diameter 2. We have \( L = \emptyset \) and \( \{w\} \) is an \( H \)-dominating set for all \( u \neq w \neq v \). If \( DOM = \{w\} \), then Algorithm 2-vs-4 executes exactly one BFS (from \( w \)) which has depth 1 which disproves the claim. Note that his proof works for all \( s \leq n/2 \).