Local Stable Marriage with Strict Preferences

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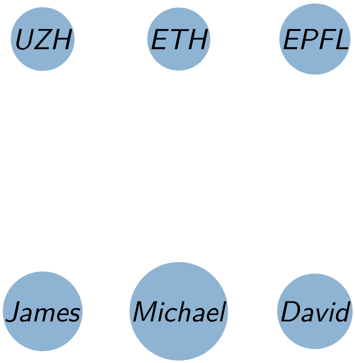
Seminar of Distributed Computing

Mar. 12, 2014

Simple example



Preferenes Table

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James	UZH>ETH>EPFL
Michael	ETH>EPFL>UZH
David	EPFL>ETH>UZH

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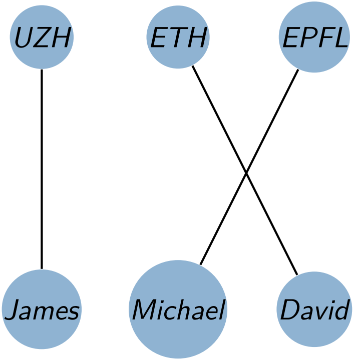

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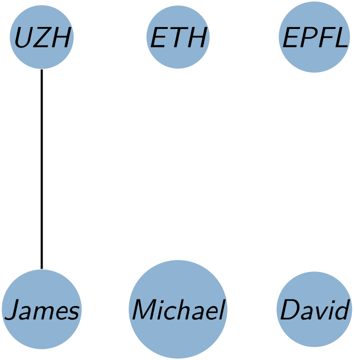

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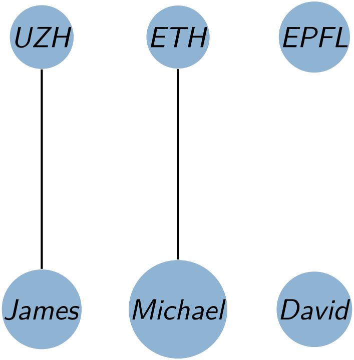

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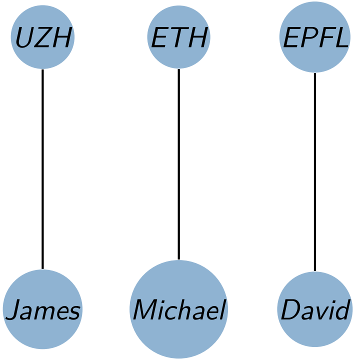

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Stable Matching

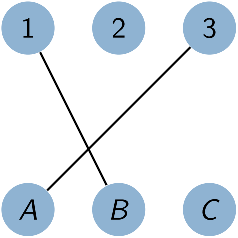
Definition

- Given two sets of elements with their set of preferences.
- A matching is a mapping from the elements of one set to the elements of the other set.
- A matching is stable if there is no blocking pair.

Blocking pair

Definition

A blocking pair is a pair such that both strictly improve by matching to each other.



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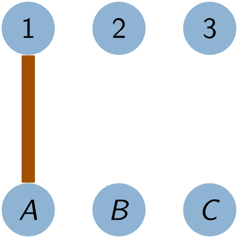

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Problem modeling

i A network matching game: (social) network: $N = (V, L)$

ii A set of vertices representing agents: V

iii A set of fixed links: $L \subseteq \{\{u, v\} | u, v \in V, u \neq v\}$

iv A set of potential matching edges:
$E \subseteq \{\{u, v\} | u, v \in V, u \neq v\}$

v correlated network game: for $\forall e \in E$,
$b_u(e) = b_v(e) = b(e) > 0$

Difference between a link and an edge:

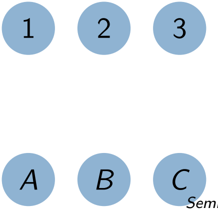
- endurable
- controllable

Local blocking pair

Assumption: each agent can match only to partners in its 2-hop neighborhood of matching edges and links.

Definition

A local blocking pair is a blocking pair of agents that are at hop distance at most 2 in the network.



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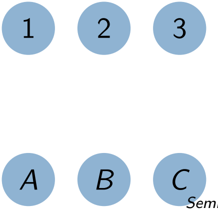

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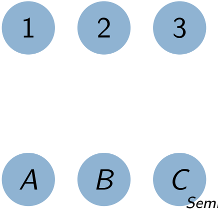

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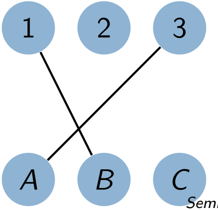

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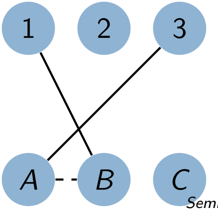

Locally stable matching

A local improvement step is one such step that add one local blocking pair to M and remove all edges that conflict with this new edge.

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Definition

A locally stable matching is a matching without local blocking pairs.

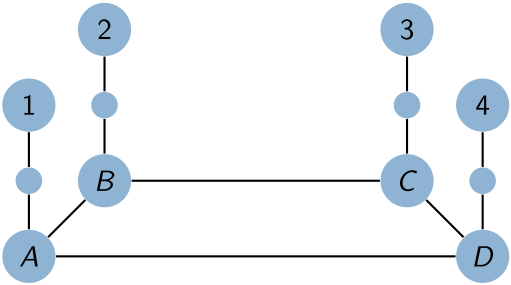
- Is it easier to find or reach using distributed dynamics than ordinary stable matchings?
- **Answer**: Locally stable matchings have a rich structure and can behave quite differently than ordinary stable matchings.

Another example

Preference-lists

v	prefrences
1	C>B>A>D
2	D>C>B>A
3	A>D>C>B
4	B>A>D>C
A	4>1>3>2
B	1>2>4>3
C	2>3>1>4
D	3>4>2>1

Circling Gadget



Explanation

Two locally stable matchings: $\{\{1, B\}, \{2, C\}, \{3, D\}, \{4, A\}\}$ and $\{\{1, C\}, \{2, D\}, \{3, A\}, \{4, B\}\}$.
Assume 1 is unmatched.

1. A is not matched with 4
   $\rightarrow$ 1 matched with A $\rightarrow$ B matched with 1 $\rightarrow$ some node unmatched

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   - 2.2 B matches with 2
   - $\rightarrow$ 2 move to C $\rightarrow$ 4 switch to B

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3. To prevent circle, one vertex must be matched to some vertex outside.
4. Existence of LSM is guarantied for the bipartite case, $\exists$ states for which REACHABILITY is not necessarily true.

Reachability

Given an instance and an initial matching, is there a sequence of local blocking pair resolutions leading to a locally stable matching?

Theorem 1

It is NP-hard to decide REACHABILITY from the initial matching $M = \emptyset$ to a given locally stable matching in a correlated network game.

Proof:

Example of 3SAT

$\rightarrow$ $(\overline{x_1} \vee x_3 \vee x_4) \wedge (x_1 \vee \overline{x_3} \vee x_5)$

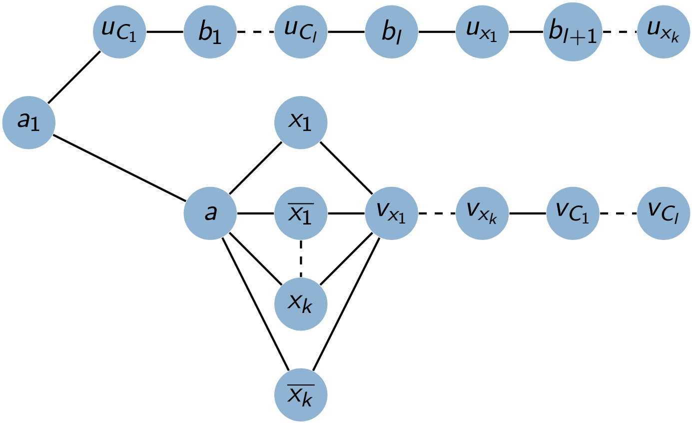
- Prove that LSM is reducible to 3SAT and vice versa
- Given a 3SAT formula with k variables $x_1, \ldots, x_k$ and l clauses $C_1, \ldots, C_l$, where clauses C_j contains the literals $l1_j$,$l2_j$ and $l3_j$.
- Divide vertices set V into two disjoint sets U and W, we have
- $U = \{u_{x_i} | i = 1...k\} \cup \{u_{C_j} | j = 1 \ldots l\} \cup \{b_h | h = 1 \ldots k + l - 1\}$,
- $W = \{v_{x_i}, x_i, \overline{x_i}, | i = 1 \ldots k\} \cup \{v_{C_j} | j = 1 \ldots l\} \cup \{a, a_1\}$.

Benefits of matching edges

$u \in U$	$w \in W$	$b(\{u, w\})$	
u_{C_j}	a	j	$j = 1, \ldots, l$
u_{x_i}	a	$i + l$	$i = 1, \ldots, k$
b_h	a	$h + 1/2$	$h = 1, \ldots, k + l - 1$
u_{C_j}	$l1_j / l2_j / l3_j$	$k + l + 1$	$j = 1, \ldots, l$
u_{x_i}	$x_i / \overline{x_i}$	$k + l + 1$	$i = 1, \ldots, k$
u_{C_j}	v_{x_i}	$k + l + 1 + i$	$i = 1, \ldots, k, j = 1, \ldots, l$
u_{x_i}	$v_{x'_i}$	$k + l + 1 + i'$	$i = 1, \ldots, k, i' = 1, \ldots, i$
u_{C_j}	$v_{C'_j}$	$2k + l + 1 + j'$	$j = 1, \ldots, k, j' = 1, \ldots, i$

Goal: reach $M^* = \{\{u_s, v_s\} | s \in \{x_1, \ldots, x_k\} \cup \{C_1, \ldots, C_l\}\}$

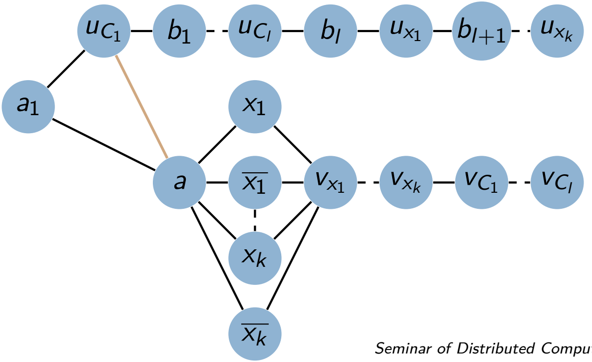
3*SAT* Gadget


u_{C_1}
b_1
u_{C_l}
b_l
u_{x_1}
b_{l+1}
u_{x_k}
a_1
x_1
a
$\overline{x_1}$
v_{x_1}
v_{x_k}
v_{C_1}
v_{C_l}
x_k
$\overline{x_k}$

3SAT → Local Stable Matching

Assume $3SAT$ is satisfiable.

$3SAT$ Gadget



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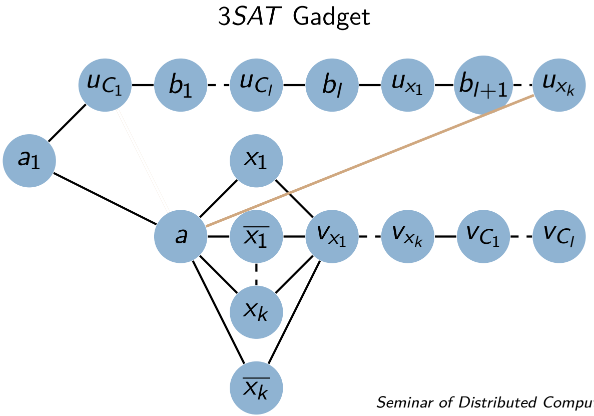

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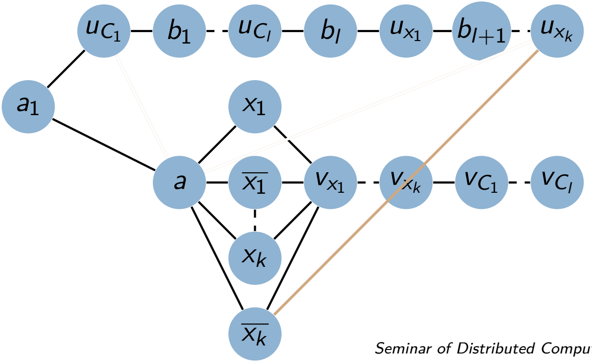


3SAT → Local Stable Matching

Step 1
First introduce $\{u_{C_j}, a\}$.

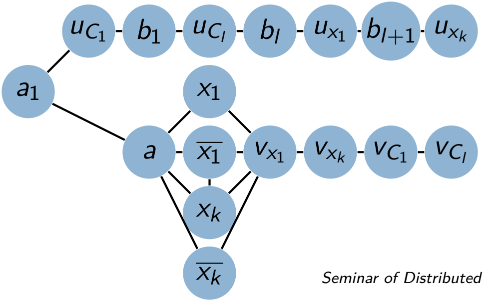
Step 2
Move it over the u-and b-vertices to u_{x_k}.

Step 3
Move it to negates its value in the satisfying assignment.

- Every clause is fulfilled
- All the clause u-vertex from a is not blocked by matching edges of variable u-vertex.
- Bypass the existing edges to reach final positions at M^*.
- Variable-edges can leave the branching to move to final position.

Local Stable Matching → 3SAT

- Assume we can reach M^* from $\emptyset$.



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- Assume we can reach M^* from $\emptyset$.
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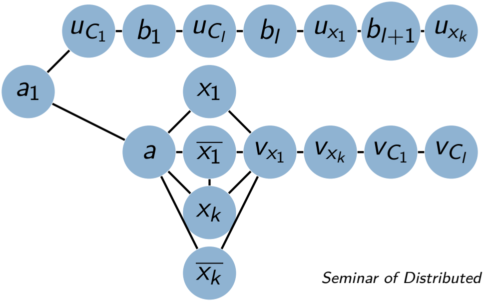

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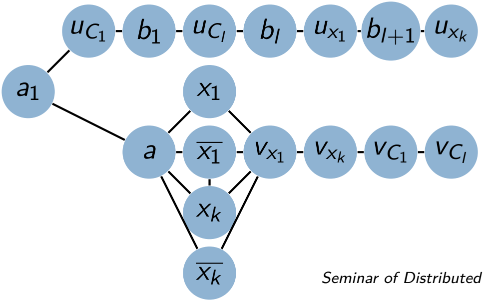

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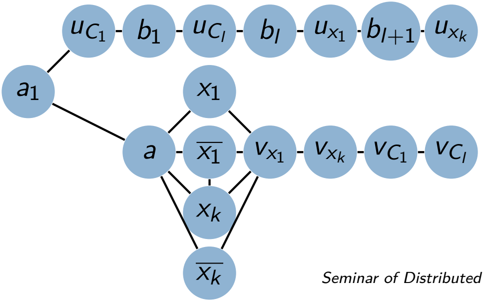
- All variable-edges have to wait at some x_i or $\overline{x_i}$ until the clause-edges have passed.



Local Stable Matching → 3SAT

From a, vertex u_{x_i} is only blocking out a different variable.

A vertex u_{C_j} will move from a if it can reach one of its literals.

All clauses can bypass the variables → there was one of its literals left open for passage.

Set each variable to the value that yields the passage for clause-edges in the branching.

Length of Sequences

Definition

The number of improvement steps required to reach locally stable matchings.

- Consider the number of improvement steps required to reach locally stable matchings.
- In general, we need an exponential number of steps before reaching LSM.
- In contrast, LSM can be reached in polynomial number of steps in correlated case.

Theorem 2

$\Longrightarrow$ For every network game with correlated preferences, every locally stable matching $M^* \in E$ and initial matching $M_0 \in E$ such that M^* can be reached from M^0 through local improvement steps,

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☺ there exists a sequence of at most $O(|E|^3)$ local improvement steps leading form M^0 to M^*.

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→ $r_{max} = max\{r(e)|e \in E\}$

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5. Overall bound: $|M_0| \times r_{max} \times r_{max} + |M^*| \times r_{max} \in O(|E|^3)$.

Recency Memory

With recency memory, each agent remembers the last partner he has been matched to.
☺ Interestingly, here we actually can ensure that a LSM can be reached.

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$\Longrightarrow$ For every network game with strict preference, links $L \subseteq (U \times W) \cup (W \times W)$, recency memory and every initial matching,

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☺ there is a sequence of $O(|U|^2|W|^2)$ many local improvement steps to a locally stable matching.

↪ **Preparation phase:**

1. while $\exists$ one $u \in U$ with u matched and u part of a blocking pair
   - allow u to switch to the better partner.
2. Terminates at most $|U| \times |W|$ steps.

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↪ **Memory phase**

1. while $\exists$ a $u \in U$ with u part of a blocking pair

   Loop

   pick u and execute a sequence of local improvement steps

   Until u is not part of any blocking pair anymore.
2. For every edge $e = \{u', w\}$ with $u' \neq u$ that was deleted during the sequence, recreate e from the memory of u'.

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- → Where every round consists of at most $|W|$ steps by u and at most $|U| - 1$ edges reproduced from memory.

Independent Set

A set of vertices in a graph, no two of which are adjacent.

Question: what is the maximal size of target locally stable matchings?

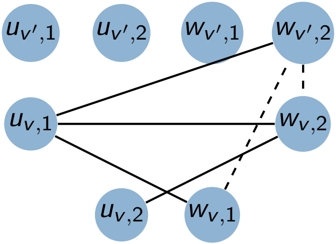
Theorem 4

Job-market game

The vertices of U are isolated in N.

For every graph $G = (V, E)$ there is a job-market game that admits a maximum locally stable matching of size $|V| + k$ if and only if G holds a maximum independent set of size k.

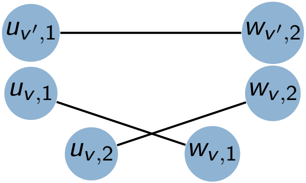
Maximum independent set → LSM



- Each $u_{v,1}$ prefers $w_{v,2}$ to every $w_{v',2}$, $v' \in N(v)$, and every $w_{v',2}$ to $w_{v,1}$.
- Each $w_{v,2}$ prefers $u_{v,1}$ to every $u_{v',1}$, $v' \in N(v)$, and every $u_{v',2}$ to $u_{v,2}$.

Claim: G has a maximum independent set of size k iff N has a locally stable matching of size $n+k$.

Proof cont.



- S is a maximum independent set in G.

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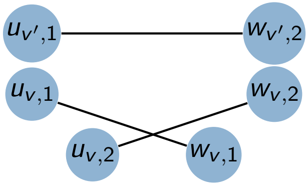

- S is a maximum independent set in G.
- $M = \{\{u_{v,1}, w_{v,2}\} | v \in V \setminus S\} \cup \{\{u_{v,1}, w_{v,1}\}, \{u_{v,2}, w_{v,2}\} | v \in S\}$.

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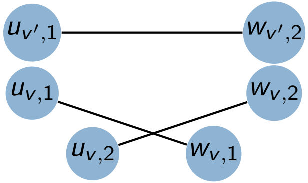

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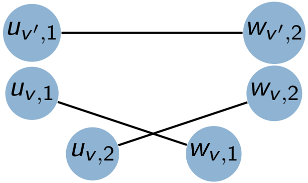
- For $v \in S$ all vertices $v' \in N(S)$ generate stable edges $\{u_{v',1}, w_{v',2}\}$ that keep $u_{v,1}$ from switching to $w_{v',2}$.

Proof cont.



- S is a maximum independent set in G.
- $M = \{\{u_{v,1}, w_{v,2}\} | v \in V \setminus S\} \cup \{\{u_{v,1}, w_{v,1}\}, \{u_{v,2}, w_{v,2}\} | v \in S\}$.
- For $v \in S$ all vertices $v' \in N(S)$ generate stable edges $\{u_{v',1}, w_{v',2}\}$ that keep $u_{v,1}$ from switching to $w_{v',2}$.
- Thus $\{u_{v,1}, w_{v,1}\}$ is stable and $w_{v,2}$ cannot see $u_{v,1}$ which stabilizes $\{u_{v,2}, w_{v,2}\}$.

LSM → Maximum independent set

- Chose M that every $u_{v,1}$ is matched.
  - Replacing partner of $w_{v,2}$ by $u_{v,1}$.

LSM $\rightarrow$ Maximum independent set

- Chose M that every $u_{v,1}$ is matched.
  - Replacing partner of $w_{v,2}$ by $u_{v,1}$.
- No $u_{v,1}$ is matched to some $w_{v',2}$ with $v \neq v'$
  - $\rightarrow$ Otherwise $u_{v,1}$ and $w_{v,2}$ can see each other and constitute a blocking pair.

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- For $S = \{v | u_{v,2} \in M\}, |S| = |M| - n$ and S is an independent set
- Every $u_{v,2}$ can only be matched to $w_{v,2}$, $u_{v,1}$ must be matched to $w_{v,1}$.
- It is stable if every $w_{v',2}$, $v' \in N(v)$, is blocked by $u_{v',1}$. Hence for every $v \in S$, $N(v) \cap S = \emptyset$.

Conclusion

- Although existence of LSM is guaranteed, but rechability is NP-hard to decide.
- In correlated network, every locally stable matching can be reached in polynomial time.
- With recency memory, reachability is guaranteed.
- We approximately find maximum locally stable matchings in job-market game.

Questions?

Please

- Questions?
- Feedback?
- ...

Reference

[1] Hoefer, Martin, and Lisa Wagner. "Locally stable marriage with strict preferences." Automata, Languages, and Programming. Springer Berlin Heidelberg, 2013. 620-631.

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[3] Gale, David, and Lloyd S. Shapley. "College admissions and the stability of marriage." American Mathematical Monthly (1962): 9-15.