Maximal Matching

(Maximal set of vertex-disjoint edges)
Being greedy
Distributed algorithm

Initially, each node only knows its incident edges

Nodes exchange messages to learn more about other nodes and edges

Time = number of communication rounds
Known bounds

\[ O(\Delta + \log^* n) \]

\[ \Omega(\text{polylog}(\Delta) + \log^* n) \]

\[ \Delta \text{ Degree} \]

\[ \text{Max number of edges incident to a node} \]
Known bounds

\[ O(\Delta + \log^* n) \]

\[ \Omega(\text{polylog}(\Delta) + \log^* n) \]

\[ \log^* n := \begin{cases} 
0 & \text{if } n \leq 1 \\
1 + \log^*(\log n) & \text{if } n > 1 
\end{cases} \]
Closing the gap

\[ O(\Delta + \log^* n) \]

\[ \Omega(\text{polylog}(\Delta) + \log^* n) \]
Closing the gap

\( O(\Delta + \log^* n) \)

in general graphs

\( \Omega(\text{polylog}(\Delta) + \log^* n) \)

simpler model
anonymous, k-edge-colored

- no two edges incident to the same node share the same color
- at most, k colors
anonymous, k-edge-colored

\[ O( \Delta + \log^* k ) \]

?.

\[ \Omega( \log^* k ) \]
anonymous, k-edge-colored

tight bound for distributed maximal matching in anonymous, k-edge-colored graphs

this work
\[ \Omega( \Delta ) \]

previous work
\[ \Omega( \log^* k ) \]

\[ \therefore \quad \Omega( \Delta + \log^* k ) \]
$\Delta \leq k$

$k$ colors
degree $\Delta$

this work
$\Omega(\Delta)$

$\Delta \leq k$

$\Omega(\ k\ ) \Rightarrow \Omega(\Delta)$
Theorem 1

Let $k$ be a positive integer. A deterministic distributed algorithm that finds a maximal matching in any anonymous, $k$-edge-colored graph requires at least $k - 1$ communication rounds.

$$\Delta \leq k$$
$$\Omega(k) \Rightarrow \Omega(\Delta)$$
$$\Omega(k-1) \Rightarrow \Omega(\Delta) \Rightarrow \Omega(\Delta + \log^* k)$$
$$\Omega(\log^* k)$$
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Greedy

\[ \Delta \leq k \]
\[ \Omega( k ) \Rightarrow \Omega( \Delta ) \]
\[ \Omega( k - 1 ) \Rightarrow \Omega( \Delta ) \Rightarrow \Omega( \Delta + \log^* k ) \]
\[ \Omega( \log^* k ) \]
Deterministic distributed greedy algorithm to find a maximal matching on a k-edge-colored anonymous graph

This work \( \Omega(k - 1) \)
Distributed algorithm

radius-\(k\) neighbourhoods

Initially, each node only knows its incident edges, its radius-0 neighbourhood
Deterministic distributed greedy algorithm to find a maximal matching on a k-edge-colored anonymous graph

same radius-2 view

after 2 communication rounds
Deterministic distributed greedy algorithm to find a maximal matching on a $k$-edge-colored anonymous graph.

This greedy algorithm needs one more communication round.

$\Omega(k-1)$
Tight lower bound for deterministic distributed maximal matching on a $k$-edge-colored graph

back to this work $\Omega(k - 1)$
Local output
d-regular graph

3-regular
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Theorem 2

Let $k \geq 3$ and $d = k - 1$

Assume a distributed algorithm that finds a maximal matching in any $d$-regular $k$-color graph. Then there are two $d$-regular $k$-colored graphs $A$, $B$ such that a node $u_e$ has the same $(d - 1)$-radius view in $A$ and $B$ and $u_e$ is unmatched in $A$ and matched in $B$. 

Greedy algorithm $\Theta(k - 1)$

This work $\Omega(k - 1)$
Building a worst case

two $d$-regular $k$-colored graphs $A$, $B$ such that a node $u_e$
has the same $d$-radius view in $A$ and $B$
and $u_e$ is unmatched in $A$ and matched in $B$

$k$-colors, $d$-regular

d = k - 1
k \geq 3 \text{ and } d = k - 1

**Group**

Generators = \{ 1, 2, \ldots, k \}

Operation: concatenation

\begin{align*}
1 \cdot 3 &= 13 \\
32 \cdot 1 &= 321
\end{align*}

Identity element: \( e \)

Inverse

\begin{align*}
1 \cdot 1 &= e \\
21 \cdot 1 &= 2 \cdot e = 2 \\
342 \cdot 213 &= 3413
\end{align*}

Associativity
\[ k \geq 3 \text{ and } d = k - 1 \]

\[ \Omega(k - 1) \]

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Associativity
$k \geq 3$ and $d = k - 1$

Forbidden color

d-regular, $k$-color; $d = k - 1$

3-regular, 3-color

2-regular, 3-color
\( k \geq 3 \) and \( d = k - 1 \)

Worst case graphs

two \( d \)-regular \( k \)-colored graphs \( A, B \) such that a node \( e \) has the same \( d \)-radius view in \( A \) and \( B \) and \( u_e \) is unmatched in \( A \) and matched in \( B \)
Simplifying the graph

leveraging symmetry

same radius-$\infty$ view

3-regular, 4-color

\[ \Omega(k - 1) \]
Simplifying the graph

leveraging symmetry

same radius-∞ view
Templates

3-regular, 4-color
Templates
Templates
Templates
Templates
Templates
Templates

\[ \Omega(k - 1) \]
Incompatible outputs

\[ \Omega( k - 1 ) \]

\[ k = 4, \; d = k - 1 = 3 \]

same radius-2 view
Induction

Two degree-i templates such that a root node
- produces different outputs;
- radius-(i - 1) neighbourhoods are identical

i = 1: base case
i > 1: by induction
i = d = k - 1: result
Base case

\[ \Omega(k - 1) \]
Base case

Ω( k - 1 )

this work
Base case

\[ \Omega(k - 1) \]

degree 1 templates, same radius-0 view, different output
Base case

$\Omega(k - 1)$

Diagram: Two sets of objects labeled with $x$, $y$, and $z$. The objects are connected by lines labeled $x$, $y$, and $z$. The diagram on the right has an additional line connecting $x$ and $z$.
Inductive step

Ω(k - 1)
Inductive step

\[ \Omega(k - 1) \]
Inductive step

degree-2 templates, same radius-1 view, different output

this work
\[ \Omega(k - 1) \]
degree-3

this work

$\Omega(k - 1)$
degree-3

this work

\(\Omega(k - 1)\)
degree-3

$\Omega(k - 1)$

degree-2 templates, same radius-2 view, different output
Theorem 2

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anonymous, k-edge-colored

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this work

\[ \Omega(\Delta) \]

previous work

\[ \Omega(\log^* k) \]

\[ \therefore \quad \Omega(\Delta + \log^* k) \]
Distributed maximal matching, Greedy is optimal in anonymous, k-edge-colored graphs

$\Theta(\Delta + \log^* k)$