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Distributed Computing

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Principles of Distributed Computing
Exercise 8: Sample Solution

1 Coloring Rings

a) Let $n \geq 4$ be even, and $r = n/2 - 2$. Let R_n be the set of all labeled rings on n vertices (there are $(n-1)!/2$ of those); since a correct algorithm has to produce a valid coloring for each of those graphs, we need to consider all of them. Consider the r-neighborhood graph $\mathcal{N}_r(R_n)$ of R_n. Note that for $r = n/2 - 2$ the r-neighborhood of a node contains all but three identifiers, ordered according to their occurence.

There exists a correct algorithm to legally color an n-vertex ring with two colors in r rounds if and only if $\mathcal{N}_r(R_n)$ is bipartite, i.e., the r-neighborhood graph contains no odd cycle. However, there is one of length $n-1$:

$$(1, \ldots, n-3), (2, \ldots, n-2), (3, \ldots, n-1), (4, \ldots, n-1, 1), (5, \ldots, n-1, 1, 2), \ldots,$$
$$(n-1, 1, 2 \ldots, n-4), (1, \ldots, n-3)$$

Note that not all of these r-neighborhoods exist in a single ring on n vertices. However, the following n-vertex rings together contain all of the above r-neighborhoods, where "..." signifies labels in ascending order, and $1 \leq k \leq \lceil \frac{n}{3} \rceil - 1$:

$$(1, \ldots, n)$$
$$(1, 2, 3, n, 4, \ldots, n-1)$$
$$\vdots$$
$$(1, \ldots, 3k, n, 3k+1, \ldots, n-1)$$
$$\vdots$$

The second ring is the ring for $k = 1$, and was included for illustration.

Since there is an odd cycle in $\mathcal{N}_r(R_n)$ for $r \leq n/2 - 2$, no algorithm that correctly colors every even ring of length n with 2 colors in at most $n/2 - 2$ rounds can exist.

b) Each node informs its two neighbors whether it is in the MIS or not and additionally sends its identifier. If node v is in the MIS, it sets its color to 1. If v is not in the MIS but both of its neighbors are, then v sets its color to 2. If v has a neighbor w not in the MIS, v chooses color 2 if its identifier is larger than w's identifier, otherwise v chooses the color 3.

The algorithm only needs one communication round. Correctness follows from the fact that either a node v is in the MIS or at least one of its neigbhors is. Thus, a MIS can at best be computed one round faster than a 3-coloring, which implies that computing a MIS costs at least $(\log^* n)/2 - 2$ rounds.

2 Ramsey theory

a) Let us fix the edge-color blue for knowing each other and the edge-color red for not knowing each other. We can also assume $p \geq 2$, else already one node would violate the condition. If any group of three people must contain at least one person that does not know the other two, then there can be no blue path of length two (else one would take the three nodes from the path and violate the condition).

If we take $2p - 2$ nodes, then we can build a graph that satisfies our conditions: First maximize the number of blue paths of length one, resulting in a perfect matching where every node is connected with a blue edge to exactly one other node. We now have $p - 1$ pairs of people that mutually know each other via a blue edge. Now let us color all other edges in red. Now at most $p - 1$ people mutually do not know each other: If p people would mutually not know each other, then by the pigeonhole principle one blue edge would need to be removed.

Now let us look at $2p - 1$ nodes: Assume there are at most $p - 1$ people that mutually do not know each other, because else we would have p people mutually not knowing each other. Let us look at any such set of $p - 1$ people that mutually do not know each other: we cannot add any of the remaining at least (2p-1)-(p-1)=p nodes, because each of them knows exactly one node of the set. But again, with the pigeonhole principle, one node in the set of size at most (p-1) must know at least two nodes from the other set of size at least p, violating the condition.

In summary, this means that there is a solution for $2p - 2$ nodes but not for $2p - 1$ nodes, giving us a sharp bound.