



# Principles of Distributed Computing

## Exercise 10: Sample Solution

### 1 Determining the Median

As stated in the hint, we start with initializing the nodes to give them IDs from  $1, \dots, n$ . Then, the node with ID  $i$  transmits its token in time slot  $i$ . Each node uses two integer variables  $n^-$ ,  $n^+$  to count how many other nodes have a token that is smaller/bigger than its own token. Once a node has received the tokens of all other nodes, it just needs to check if  $n^+ = n^-$ . The node whose token is the median can simply broadcast it afterwards.

Let us briefly analyze the time and space used by this. We know from the lecture that initialization takes  $\mathcal{O}(n)$  rounds. Sending all tokens afterwards takes another  $n$  rounds, i.e., does not increase the asymptotic runtime.

Each node needs to store a new ID of up to  $n$ , which takes up  $\mathcal{O}(\log n)$  space. The variables  $n^-$ ,  $n^+$  from the algorithm can also be stored by using  $\mathcal{O}(\log n)$ . Note that we do not need to store the values of the tokens of other nodes. Therefore, we only need  $\mathcal{O}(\log n)$  space in total.

The correctness follows by the construction of the algorithm. The unique node whose upper and lower counter has the same value can broadcast its token and thus all nodes are aware that it is the median.

**Remark:** We can further reduce the memory usage if, instead of storing  $n^+$  and  $n^-$ , we only store the difference  $n^+ - n^-$  in a variable called  $\eta$ . Every time a node receives a token which is bigger than its own token, it increments  $\eta$ . If a node receives a token smaller than its own token, it decrements  $\eta$ . The median of all tokens will then be found in the node which has  $\eta = 0$ .

### 2 Maximum

See Algorithm 1 below for a pseudocode description of the algorithm. We will now proceed to analyze the runtime of this algorithm.

A round is *good* if at least half the vertices exit in that round. Observe that at most  $\log(n)$  good rounds are needed to find the maximum. Since the leaders are chosen randomly and independently in each round, a round is good with probability at least  $\frac{1}{2}$ . Let the random variable  $G$  be the number of good rounds after  $4c \cdot \log(n)$  rounds, for  $c \geq 1$ . The expectation of  $G$  is bounded by

$$E[G] \geq \frac{1}{2}4c \cdot \log(n) = 2c \cdot \log(n) \geq 2 \cdot \log(n)$$

Using the Chernoff bound and setting  $\delta = \frac{1}{2}$  we get the following as probability of failure (i.e., not having enough good rounds):

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**Algorithm 1** Algorithm for determining the maximum

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1: while TRUE do
2:   elect leader
3:   leader broadcasts value
4:   if own value is bigger then
5:     broadcast own value
6:   end if
7:   if no transmitter then
8:     leader has max
9:   else if single transmitter then
10:    transmitter has max
11:  else if own value  $\leq$  leader then
12:    exit
13:  end if
14: end while
```

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$$\begin{aligned} Pr[G < \log(n)] &= Pr[G < (1 - \delta)2 \cdot \log(n)] \\ &\leq Pr[G < (1 - \delta)E[G]] \\ &\leq e^{-\frac{\delta^2}{2}E[G]} \\ &\leq e^{-\frac{1}{4}c \log(n)} \\ &= n^{-\frac{c}{4}} \end{aligned}$$

Additionally, we have a failure probability of at most  $n^{-c'}$  every time we elect a leader. Note that we look at the probability of succeeding in  $4c \cdot \log(n)$  rounds, and we perform one leader election in each of these rounds. We set  $c' = \frac{c}{4}$ ; this is possible, since the notion of w.h.p. allows us to choose the constant  $c'$  in the exponent of the success probability.

With the union bound we can derive

$$\begin{aligned} Pr[\text{fail}] &= P \left[ \{G < \log(n)\} \cup \bigcup_i \{\text{leader election in round } i \text{ failed}\} \right] && | \text{ Union Bound} \\ &\leq n^{-c'} + 4c \cdot \log(n)n^{-c'} \\ &= (1 + 4c \cdot \log(n))n^{-c'} && | n > 1, c \geq 1 \\ &\leq 5c \cdot \log(n)n^{-c'} \\ &= 2^{\log(5c)} \cdot \log(n)n^{-c'} \\ &< 2^{\log(5c) \cdot \log(n)} n^{-c'} \\ &= n^{-c' + \log(5c)} \\ &= n^{-\alpha} \end{aligned}$$

where  $\alpha$  is independent of  $n$  and can be set arbitrarily high by choosing the constant  $c'$ .