

Exercise 6

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1 Lower Bound for Locally-Minimal Coloring

For a graph $G = (V, E)$, a coloring $\phi : V \rightarrow \{1, 2, \dots, Q\}$ is called *locally-minimal* if it is a proper coloring, meaning that no two adjacent vertices v and u have $\phi(v) = \phi(u)$, and moreover, for each node v colored with color $q = \phi(v) \in \{1, 2, \dots, Q\}$, all colors 1 to $q - 1$ are used in the neighborhood of v . That is, for each $i \in \{1, \dots, q - 1\}$, there exists a neighbor u of v such that $\phi(u) = i$.

Exercise

- (1a) In the 4th lecture, we saw a $O(\Delta \log \Delta + \log^* n)$ -round algorithm for computing a $(\Delta + 1)$ -vertex-coloring in any n -node graph with maximum degree Δ . Use this algorithm as a black box to compute a *locally-minimal coloring* in $O(\Delta \log \Delta + \log^* n)$ rounds, in an n -node graph with maximum degree Δ .

In the remainder of this exercise, we prove a lower bound of $\Omega(\log n / \log \log n)$ on the round complexity of computing a *locally-minimal coloring*, for some graphs. We note that these graphs have maximum degree $\Delta = \Omega(\log n)$ and hence, this lower bound poses no contradiction with (1a).

For the lower bound, we will use a classic graph-theoretic result of Erdős [Erd59]. Recall that the girth of a graph is the length of its shortest cycle, and the chromatic number of a graph is the smallest number of colors required in any proper coloring of the graph.

Theorem 1 (Erdős [Erd59]) *For any sufficiently large n , there exists an n -node graph G_n^* with girth $g(G_n^*) \geq \frac{\log n}{4 \log \log n}$ and chromatic number $\chi(G_n^*) \geq \frac{\log n}{4 \log \log n}$.*

Exercise

- (1b) Prove that in any locally-minimal coloring $\phi : V \rightarrow \{1, 2, \dots, Q\}$ of a tree $T = (V, E)$ with diameter d — i.e., where the distance between any two nodes is at most d — no node v can receive a color $\phi(v) > d + 1$.
- (1c) Suppose towards contradiction that there exists a deterministic algorithm \mathcal{A} that computes a locally-minimal coloring of any n -node graph in at most $\frac{\log n}{8 \log \log n} - 1$ rounds. Prove that when we run \mathcal{A} on the graph G_n^* , it produces a (locally-minimal) coloring with at most $Q = \frac{\log n}{4 \log \log n} - 1$ colors. For this, you should use part (1b) and the fact that G_n^* has girth $g(G_n^*) \geq \frac{\log n}{4 \log \log n}$.
- (1d) Conclude that any locally-minimal coloring algorithm needs at least $\frac{\log n}{8 \log \log n}$ rounds on some n -node graph.

References

[Erd59] Paul Erdős. Graph theory and probability. *Canada J. Math*, 11:34G38, 1959.