Exercise 4

Lecturer: Mohsen Ghaffari

1 Problem 1, Lower Bound for Locally-Minimal Coloring

For a graph $G = (V, E)$, a coloring $\phi : V \to \{1, 2, ..., Q\}$ is called *locally-minimal* if it is a proper coloring, meaning that no two adjacent vertices v and u have $\phi(v) = \phi(u)$, and moreover, for each node v colored with color $q = \phi(v) \in \{1, 2, ..., Q\}$, all colors 1 to $q - 1$ are used in the neighborhood of v. That is, for each $i \in \{1, \ldots, q - 1\}$, there exists a neighbor u of v such that $\phi(u) = i$.

Exercise

(1a) In the 3^{rd} lecture, we saw a $O(\Delta \log \Delta + \log^* n)$-round algorithm for computing a $(\Delta + 1)$-vertex-coloring in any n-node graph with maximum degree Δ. Use this algorithm to compute a *locally-minimal coloring* in $O(\Delta \log \Delta + \log^* n)$ rounds, in any n-node graph with maximum degree Δ.

In the remainder of this exercise, we prove a lower bound of $\Omega(\log n/ \log \log n)$ on the round complexity of computing a *locally-minimal coloring*, for some graphs. We note that these graphs have maximum degree $\Delta = \Omega(\log n)$ and hence, this lower bound poses no contradiction with (1a).

For the lower bound, we will use a classic graph-theoretic result of Erdős [Erd59]. Recall that the girth of a graph is the length of its shortest cycle, and the chromatic number of a graph is the smallest number of colors required in any proper coloring of the graph.

Theorem 1 (Erdős [Erd59]) *For any sufficiently large n, there exists an n-node graph G^*_n with girth $g(G^*_n) \geq \frac{\log n}{4 \log \log n}$ and chromatic number $\chi(G^*_n) \geq \frac{\log n}{4 \log \log n}$.*

Exercise

(1b) Prove that in any locally-minimal coloring $\phi : V \to \{1, 2, ..., Q\}$ of a tree $T = (V, E)$ with diameter d — i.e., where the distance between any two nodes is at most d — no node v can receive a color $\phi(v) > d + 1$.

(1c) Suppose towards contradiction that there exists a deterministic algorithm $\mathcal{A}$ that computes a locally-minimal coloring of any n-node graph in at most $\frac{\log n}{8 \log \log n} - 1$ rounds. Prove that when we run $\mathcal{A}$ on the graph G^*_n, it produces a (locally-minimal) coloring with at most $Q = \frac{\log n}{4 \log \log n} - 1$ colors. For this, you should use part (1b) and the fact that G^*_n has girth $g(G^*_n) \geq \frac{\log n}{4 \log \log n}$.

(1d) Conclude that any locally-minimal coloring algorithm needs at least $\frac{\log n}{8 \log \log n}$ rounds on some n-node graph.

References

[Erd59] Paul Erdős. Graph theory and probability. *Canada J. Math*, 11:34G38, 1959.