Network Decompositions

Exercise 1: Explain how given a $(\mathcal{C}, \mathcal{D})$ weak-diameter network decomposition of graph G, we can deterministically compute a $(\Delta + 1)$-coloring of the graph in $O(\mathcal{CD})$ rounds. Here, Δ denotes an upper bound on the maximum degree of the graph, and is given to the algorithm as an input.

Exercise 2: In this exercise, we prove that every n-node graph G has a $(\mathcal{C}, \mathcal{D})$ (strong-diameter) network decomposition for $\mathcal{C} = O(\log n)$ and $\mathcal{D} = O(\log n)$. The process that we see can be viewed as a simple and efficient sequential algorithm for computing such a network decomposition.

We determine the blocks $G_1, G_2, ..., G_{\mathcal{C}}$ of the network decomposition one by one, in C phases. Consider phase i and the graph $G \setminus (\cup_{j=1}^{i-1} G_j)$ remaining after the first $i - 1$ phases which defined the first $i-1$ blocks $G_1, \ldots, G_{i-1}$. To define the next block, we repeatedly perform a ball carving starting from arbitrary nodes, until all nodes of $G \setminus (\cup_{j=1}^{i-1} G_j)$ are removed. This ball carving process works as follows:

Consider an arbitrary node $v \in G \setminus (\cup_{j=1}^{i-1} G_j)$ and consider gradually growing a ball around v, hop by hop. After the kth step, the ball $B_k(v)$ is simply the set of all nodes within distance k of v in the remaining graph. In the very first step that the ball does not grow by more than a 2 factor—i.e., after the step corresponding to the smallest value of k for which $|B_{k+1}(v)|/|B_k(v)| \leq 2$—we stop the ball growing. Then, we carve out the inside of $B_{k+1}(v)$—i.e., all nodes in $B_k(v)$— and define them to be a cluster of G_i. Hence, these nodes are added to G_i. Moreover, we remove all boundary nodes of this ball—i.e., those of $B_{k+1}(v) \setminus B_k(v)$—from the graph considered for the rest of this phase. These nodes will never be put in G_i. We will bring them back in the next phases, so that they get clustered in future phases. Then, we repeat a similar ball carving starting at an arbitrary other node v' in the remaining graph. We continue until all nodes are removed or clustered. This finishes the description of phase i. Once phase i is completed, we move to the next phase. The algorithm terminates when all nodes have been clustered.

Prove the following properties:

1. Each cluster defined in the above process has diameter at most $O(\log n)$. In particular, for each ball that we carve, the related radius k is at most $O(\log n)$.

2. In each phase i, the number of nodes that we cluster—and thus put in G_i—is at least $1/2$ of the number of nodes of $G \setminus (\cup_{j=1}^{i-1} G_j)$.

3. Conclude that the process terminates in at most $O(\log n)$ phases, which means that the network decomposition has at most $O(\log n)$ blocks.

Exercise 3 (*optional*): Develop a deterministic distributed algorithm with round complexity poly$(\log n)$ for computing a $(\mathcal{C}, \mathcal{D})$ (strong-diameter) network decomposition in any n-node network, such that $\mathcal{C} = O(\log n)$ and $\mathcal{D} = O(\log n)$.