Distributed Computing over
Communication Networks:

Maximal Independent Set

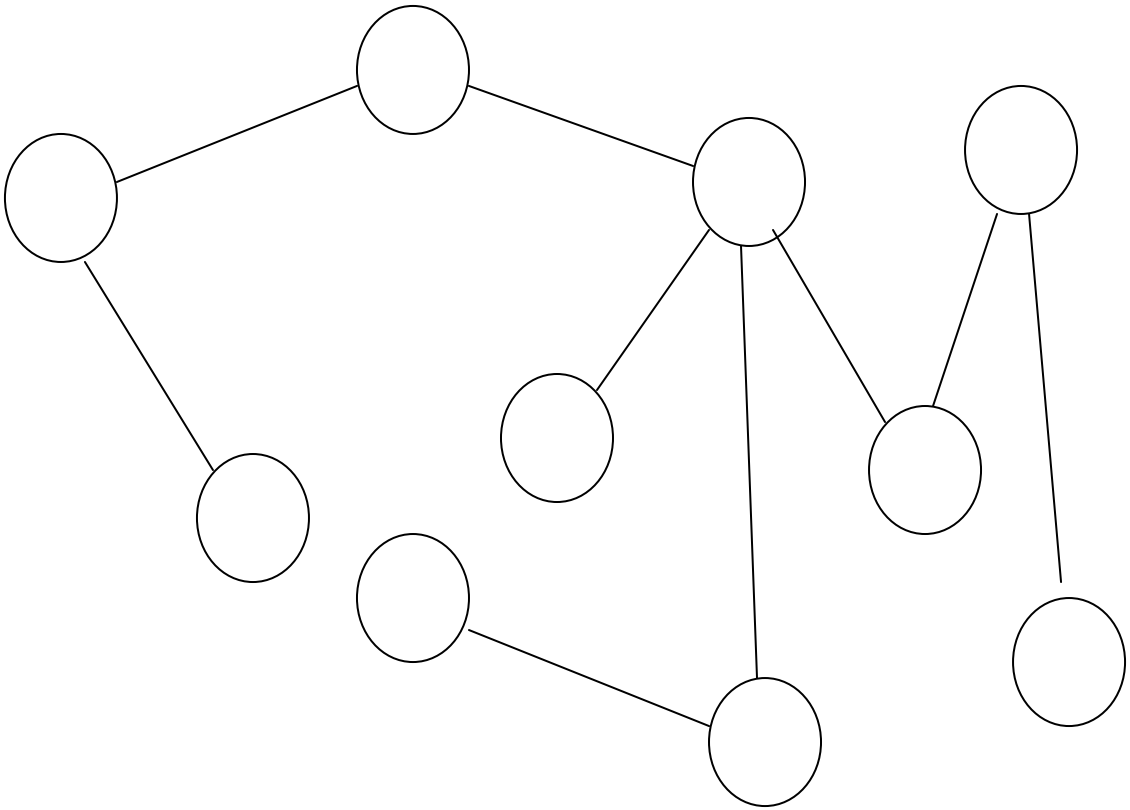
What is a MIS?

MIS

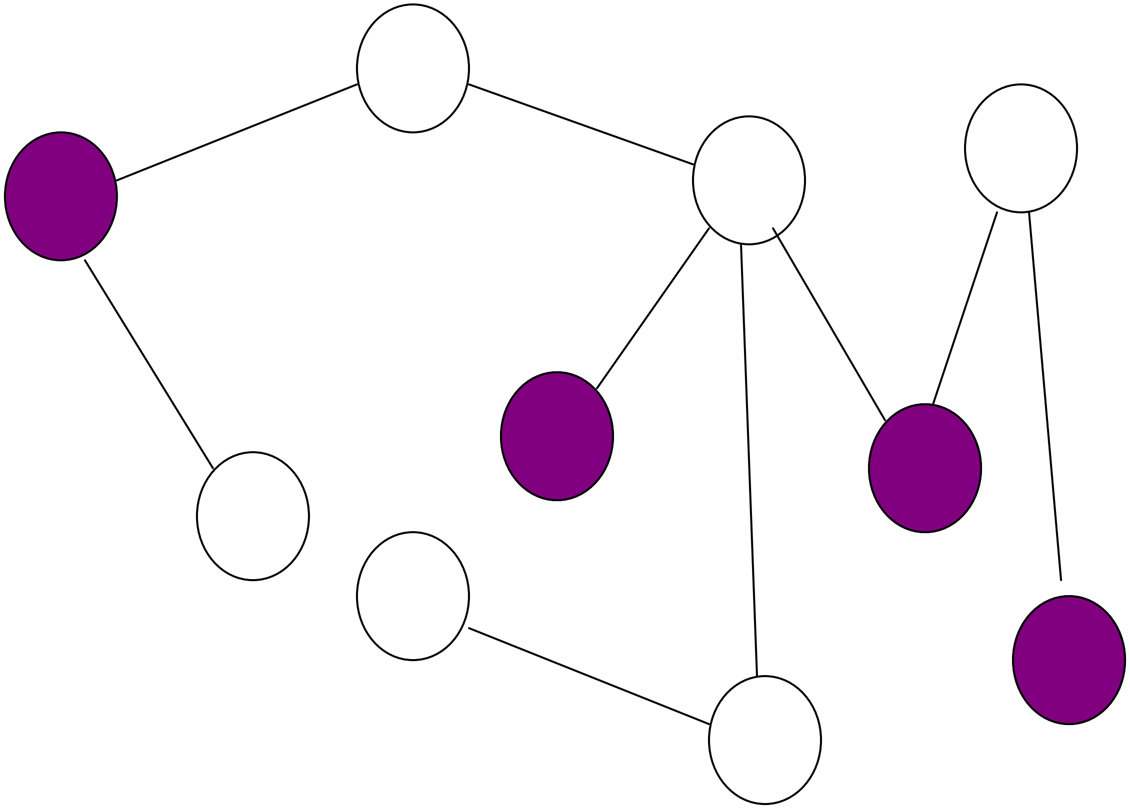
An independent set (IS) of an undirected graph is a subset U of nodes such that no two nodes in U are adjacent. An IS is maximal if no node can be added to U without violating IS (called MIS). A maximum IS (called MaxIS) is one of maximum cardinality.

... known from „classic TCS“:
applications?
backbone, parallelism, ...
complexities?

MIS and MaxIS?

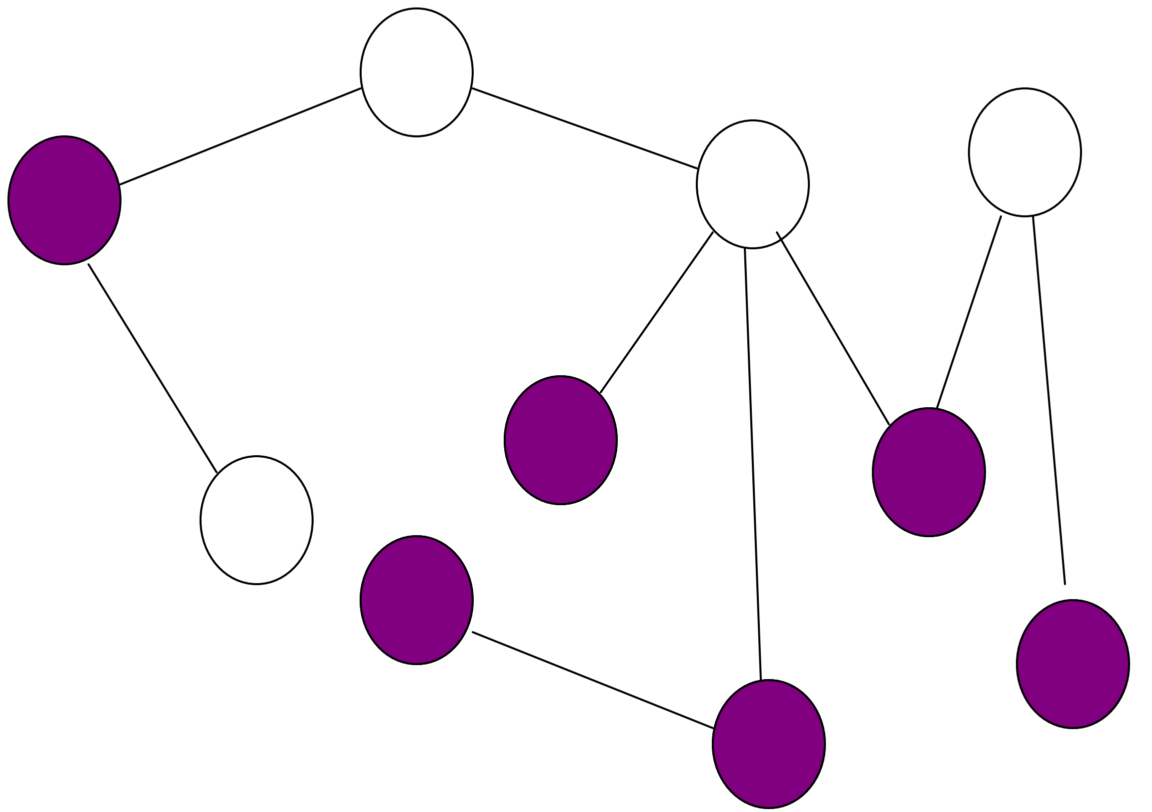


Nothing, IS, MIS, MaxIS?



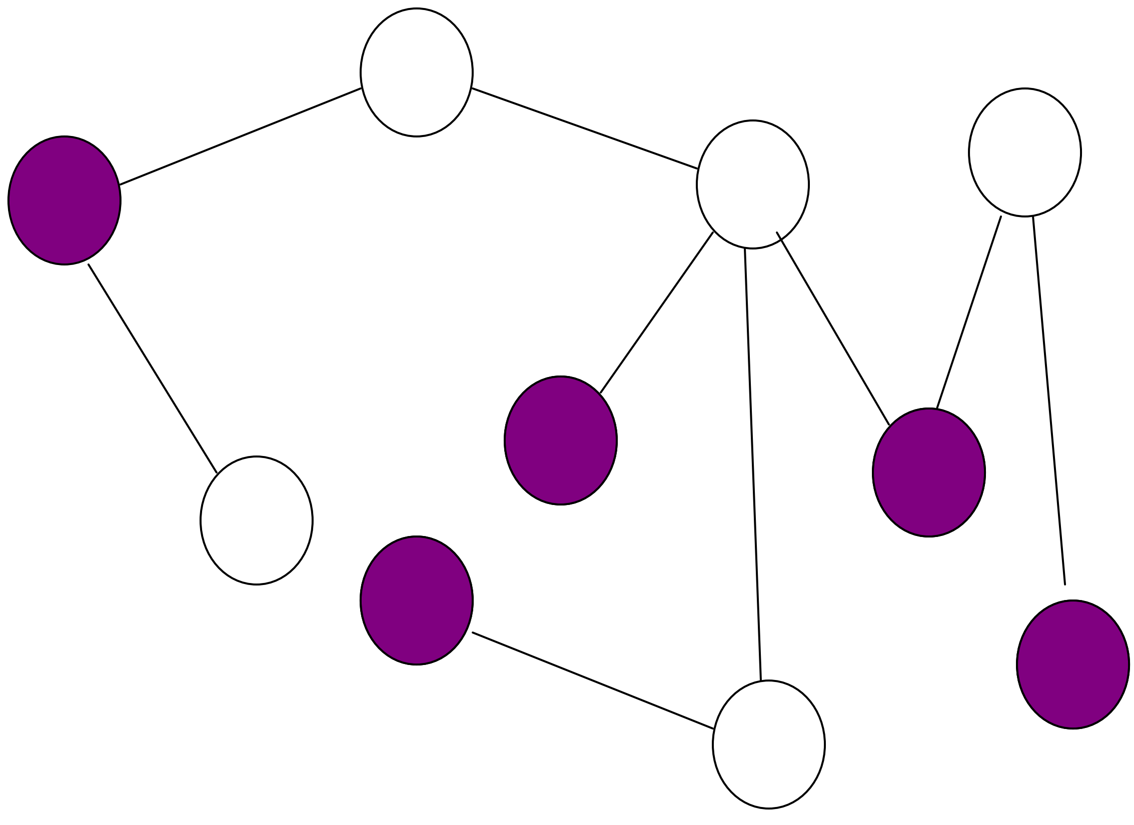
IS but not MIS.

Nothing, IS, MIS, MaxIS?



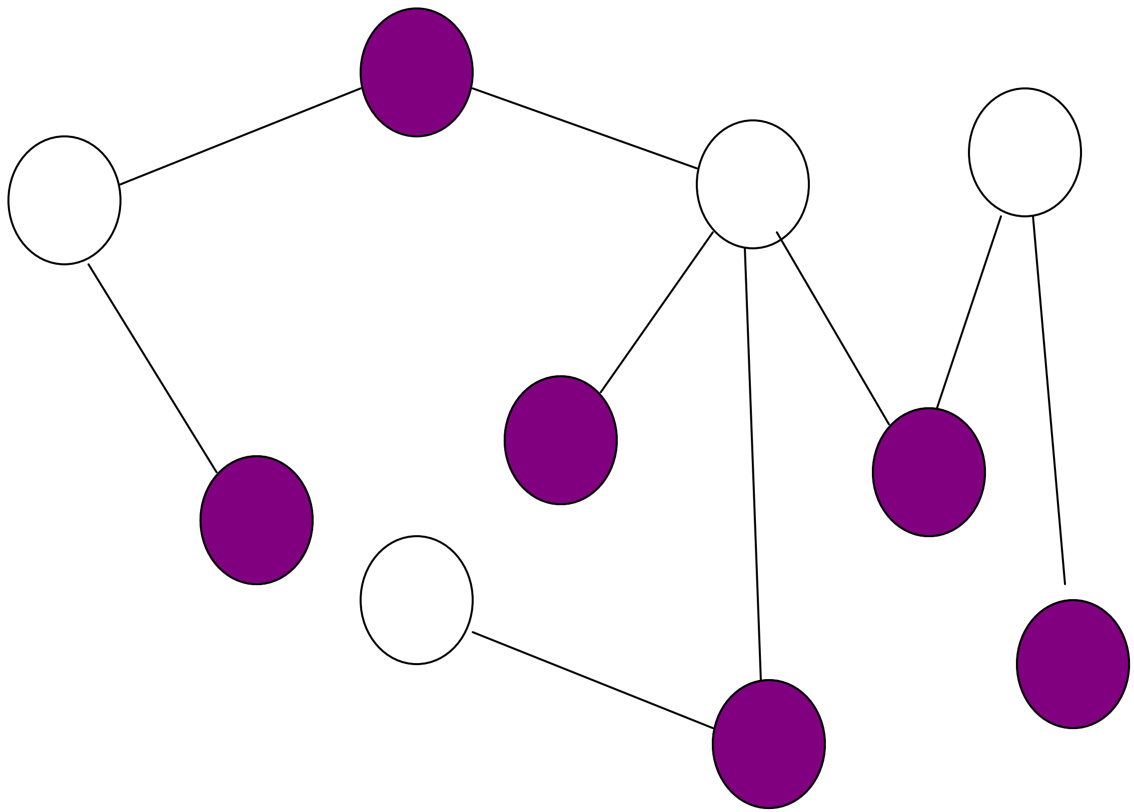
Nothing.

Nothing, IS, MIS, MaxIS?



MIS.

Nothing, IS, MIS, MaxIS?



MaxIS.

Complexities?

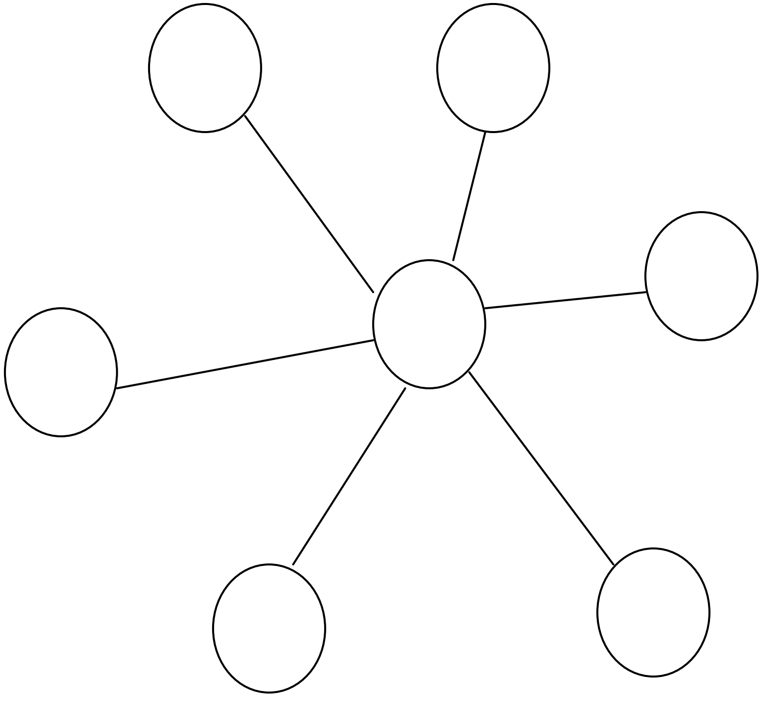
MaxIS is NP-hard!

So let‘s concentrate on MIS...

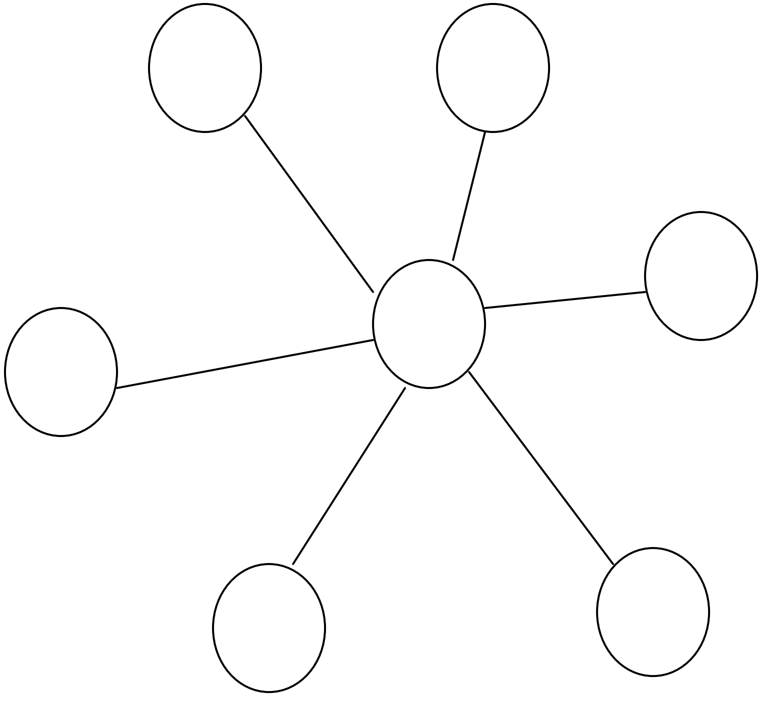
How much worse can MIS be than MaxIS?

How much worse can MIS be than MaxIS?

minimal MIS?

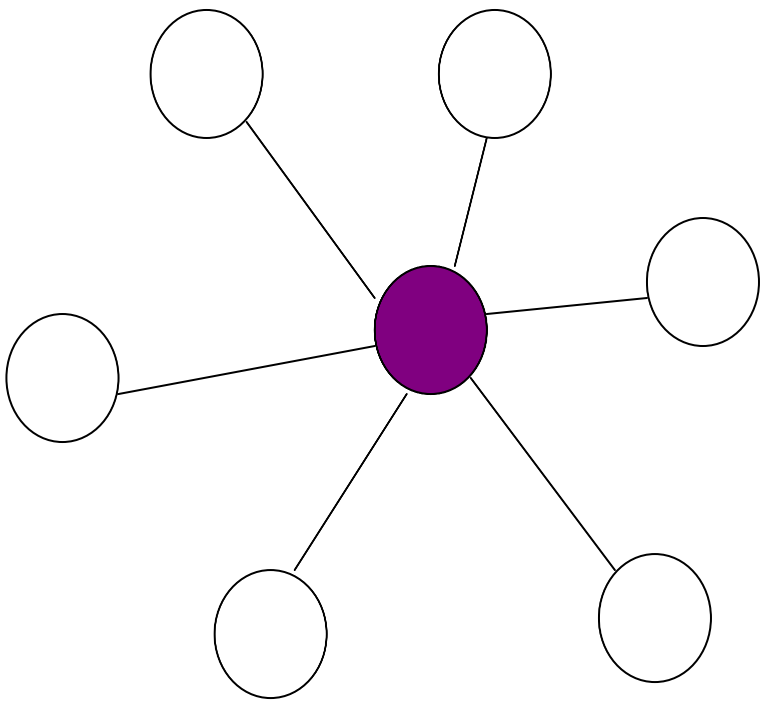


maxIS?

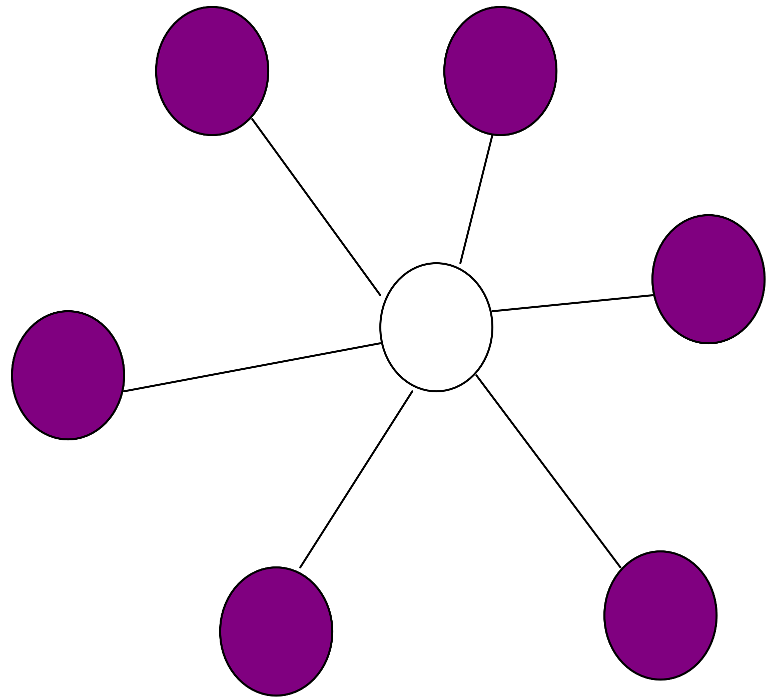


How much worse can MIS be than Max-IS?

minimal MIS?



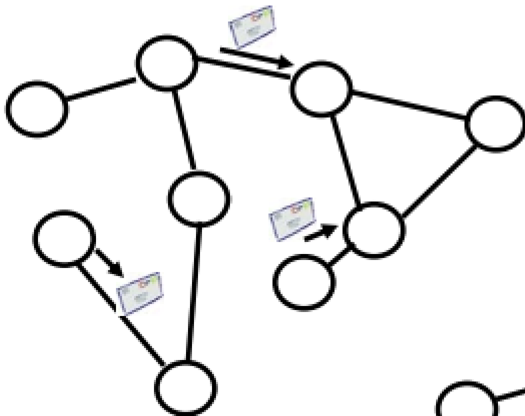
Maximum IS?



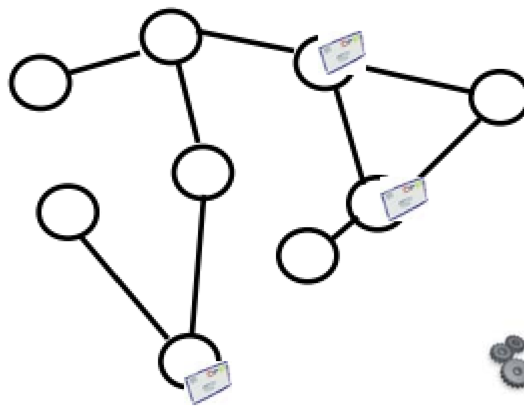
How to compute a MIS in a distributed manner?!

Recall: Local Algorithm

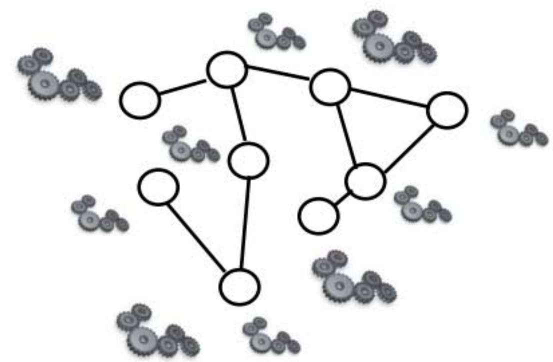
Send...



... receive...



... compute.



Slow MIS

Slow MIS

assume node IDs

Each node v:

1. If all neighbors with larger IDs have decided not to join MIS then:

   v decides to join MIS

Analysis?

Analysis

Time Complexity?

Not faster than sequential algorithm!
Worst-case example?
E.g., sorted line: $O(n)$ time.

Local Computations?

Fast! ☺

Message Complexity?

For example in clique: $O(n^2)$
($O(m)$ in general: each node needs to inform all neighbors when deciding.)

MIS and Colorings

Independent sets and colorings are related: how?

Each color in a valid coloring constitutes an independent set (but not necessarily a MIS).

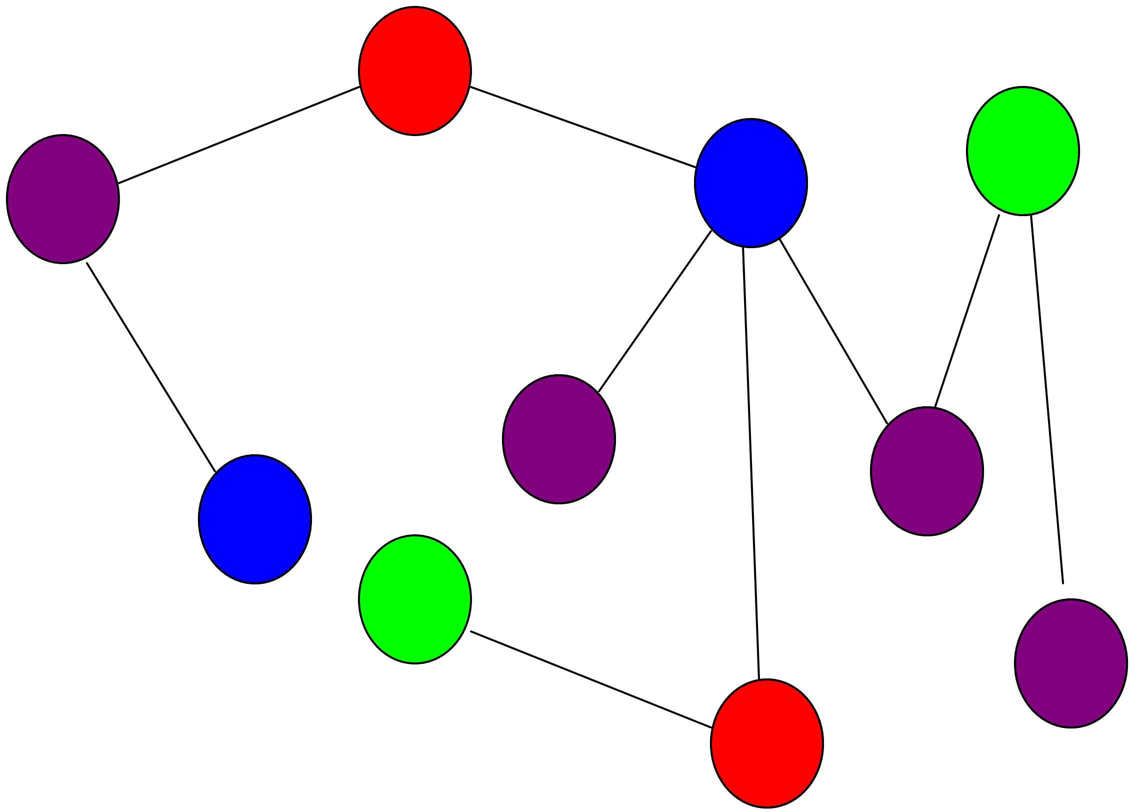
How to compute MIS from coloring?

Choose all nodes of first color. Then for any additional color, add in parallel as many nodes as possible!

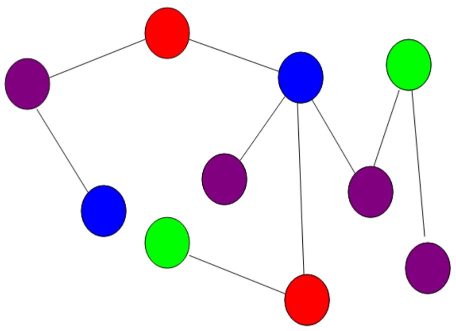
Why, and implications?

Coloring vs MIS

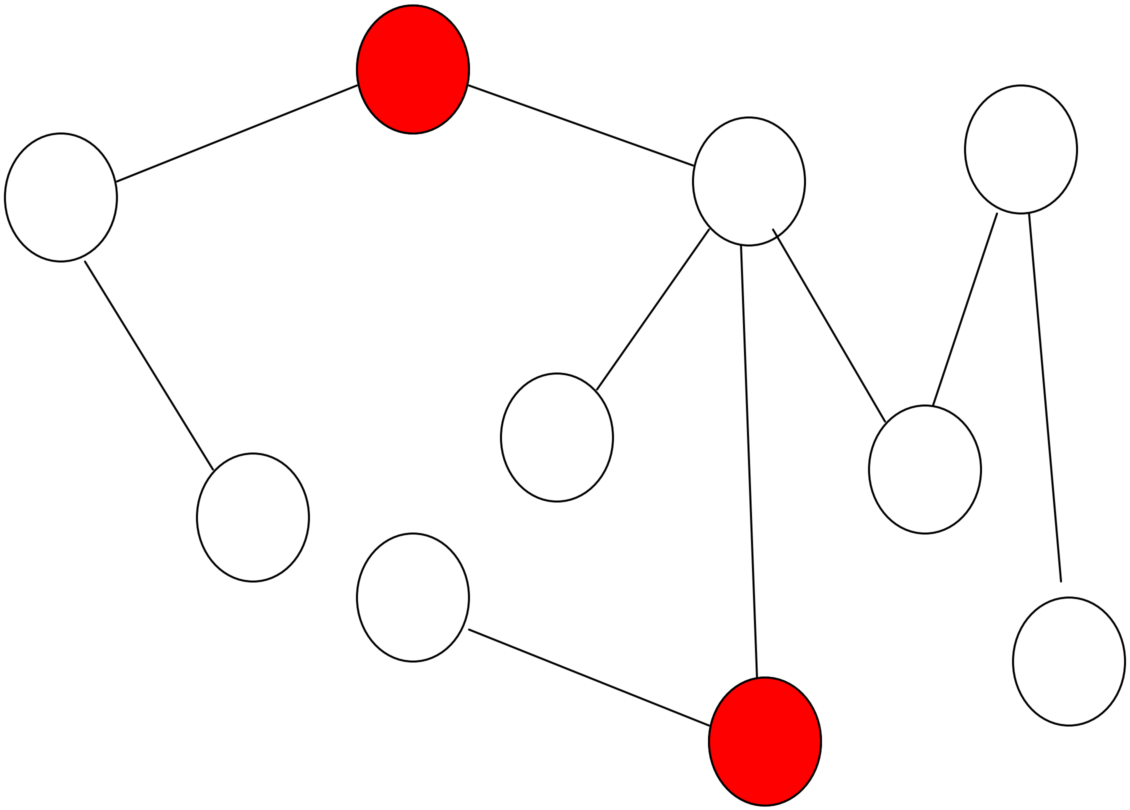
Valid coloring:



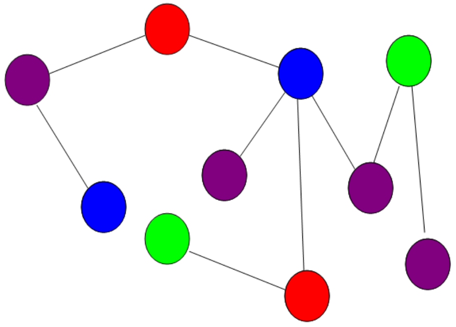
Coloring vs MIS



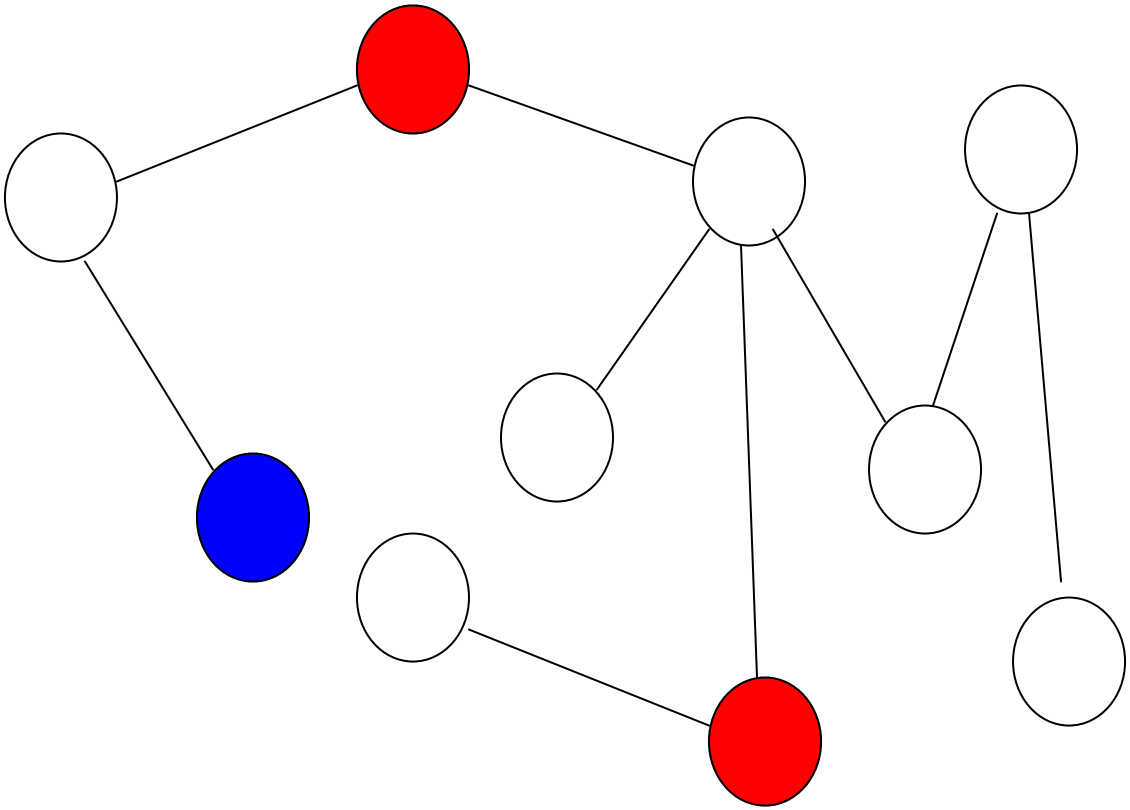
Independent set:



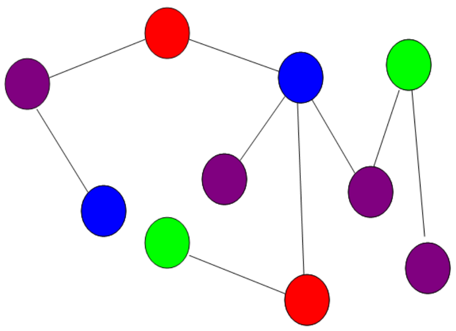
Coloring vs MIS



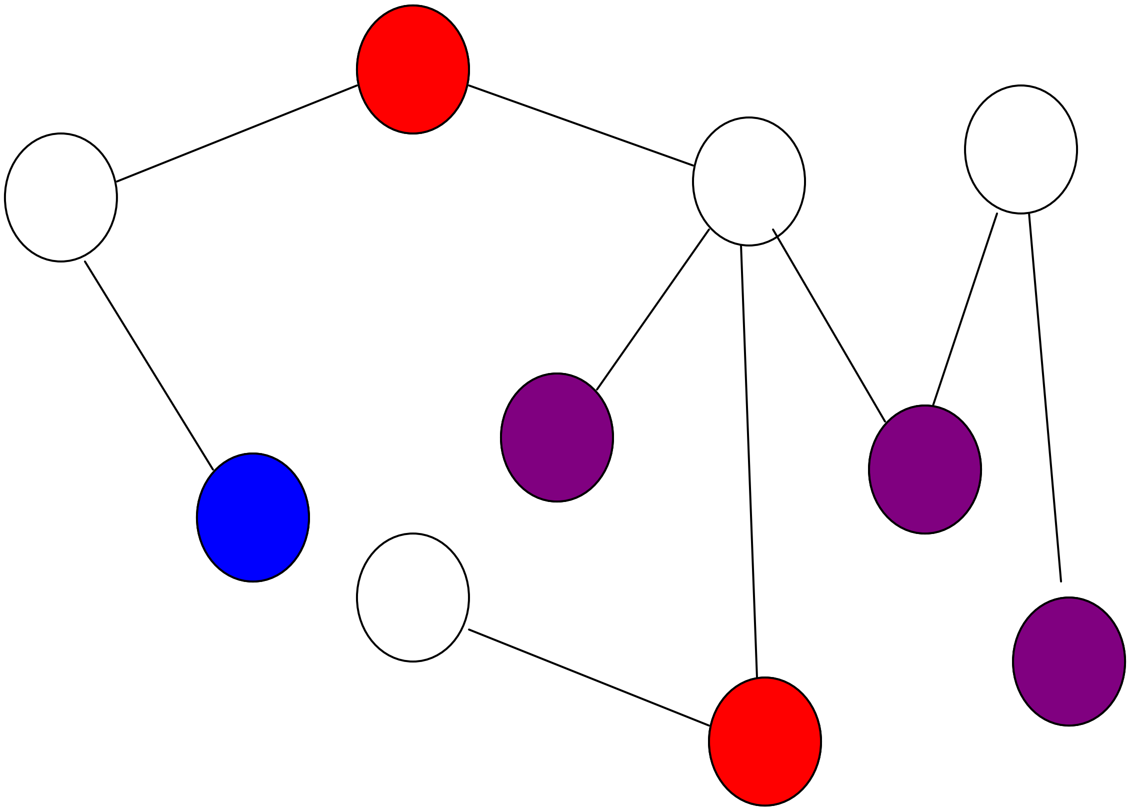
Add all possible blue:



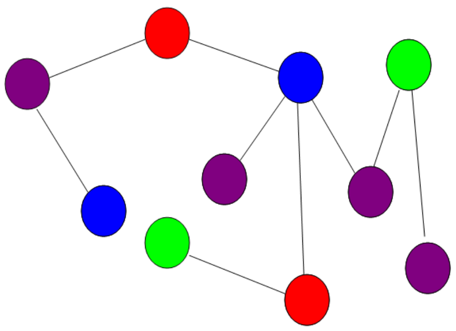
Coloring vs MIS



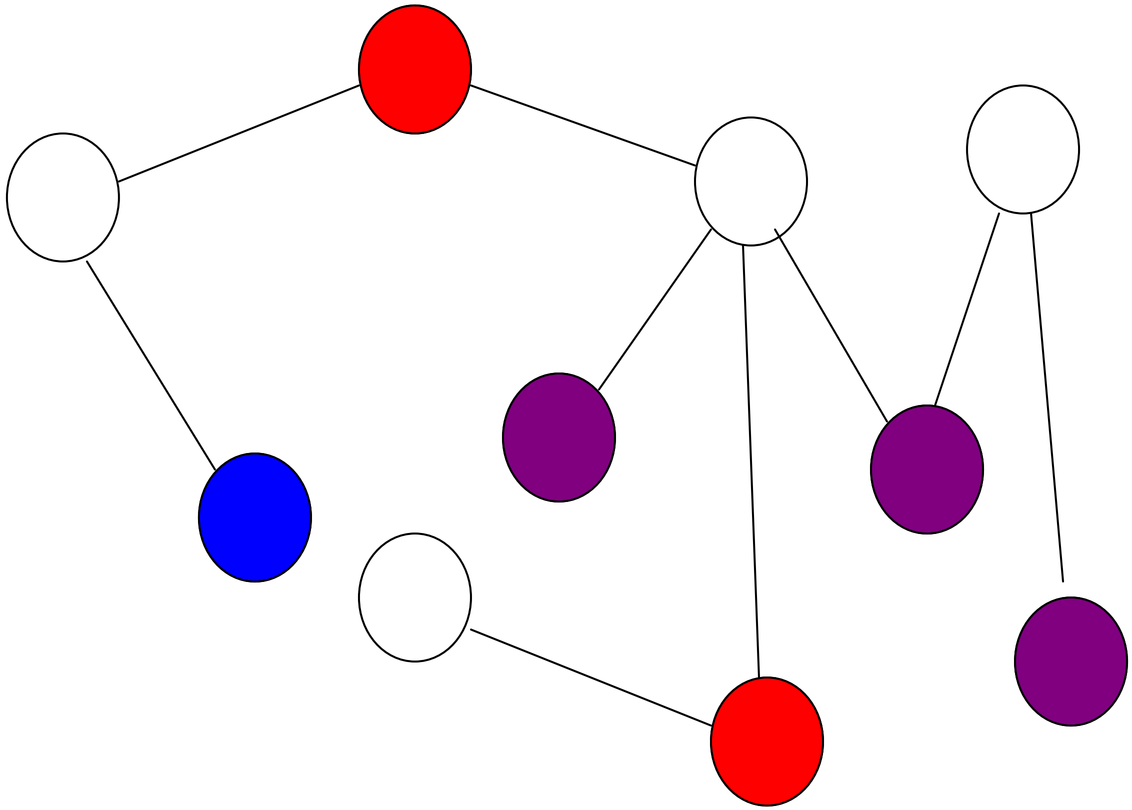
Add all possible violet:



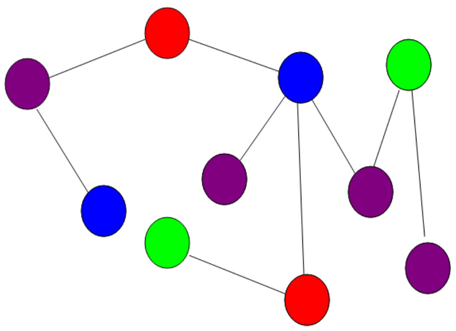
Coloring vs MIS



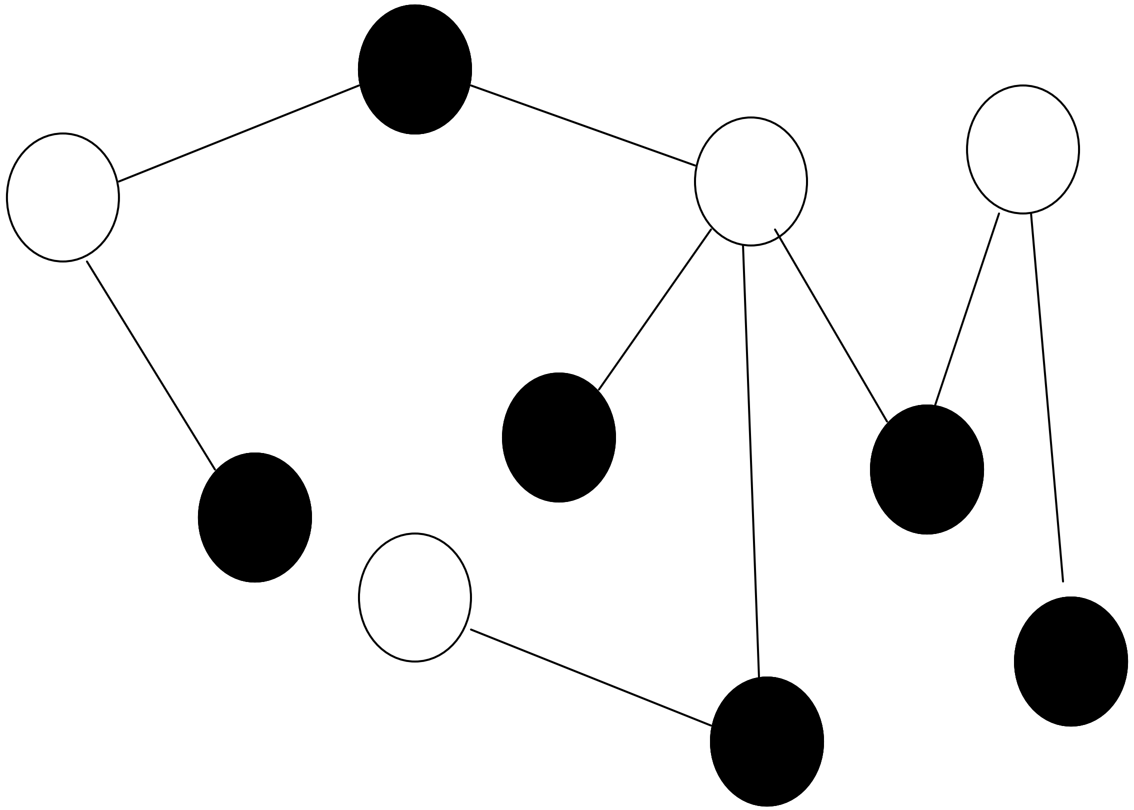
Add all possible green:



Coloring vs MIS



That's all: MIS!



Analysis of algorithm?

Why does algorithm work?

Same color: all nodes independent, can add them in parallel without conflict (not adding two conflicting nodes concurrently).

Runtime?

Lemma

Given a coloring algorithm with runtime T that needs C colors, we can construct a MIS in time C+T.

What does it imply for MIS on trees?

We can color trees in log* time and with 3 colors, so:

MIS on Trees

There is a deterministic MIS on trees that runs in distributed time O(log* n).

Better MIS Algorithms

Any ideas?

Takeaway

If you can‘t find fast deterministic algorithms, try randomization!

Ideas for randomized algorithms?

Fast MIS from 1986...

Fast MIS (1986)

Proceed in rounds consisting of phases

In a phase:

1. each node v marks itself with probability 1/(2d(v)) where d(v) denotes the current degree of v
2. if no higher degree neighbor is marked, v joins MIS; otherwise, v unmarks itself again (break ties arbitrarily)
3. delete all nodes that joined the MIS plus their neighbors, a they cannot join the MIS anymore

Why is it correct? Why IS? Why MIS?

Note: the higher the degree the less likely to mark, but the more likely to join MIS once marked!

MIS 1986

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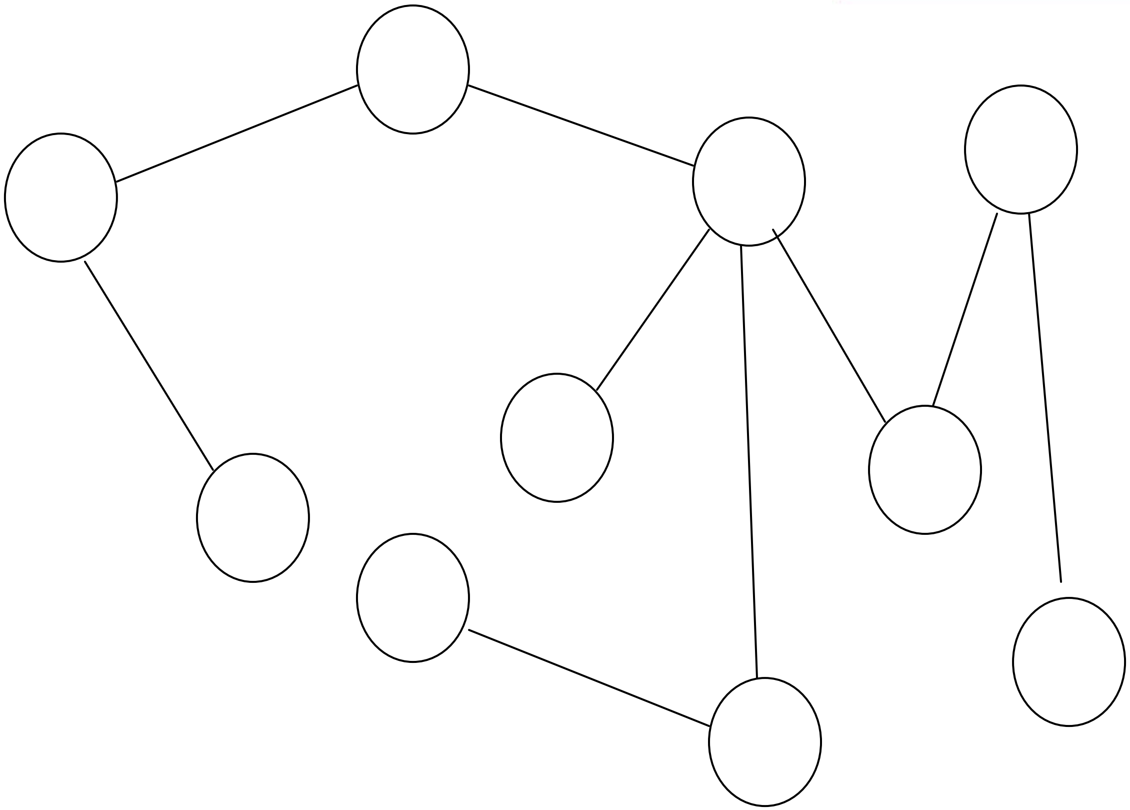
Probability of marking?



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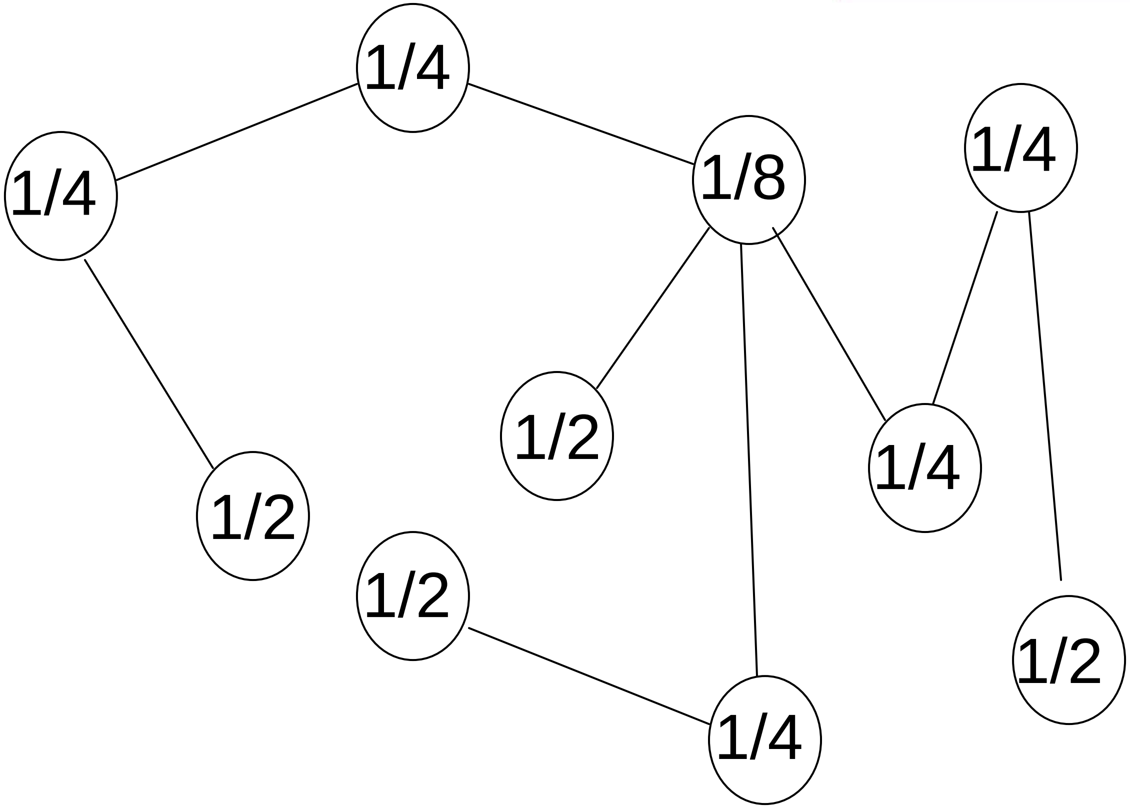

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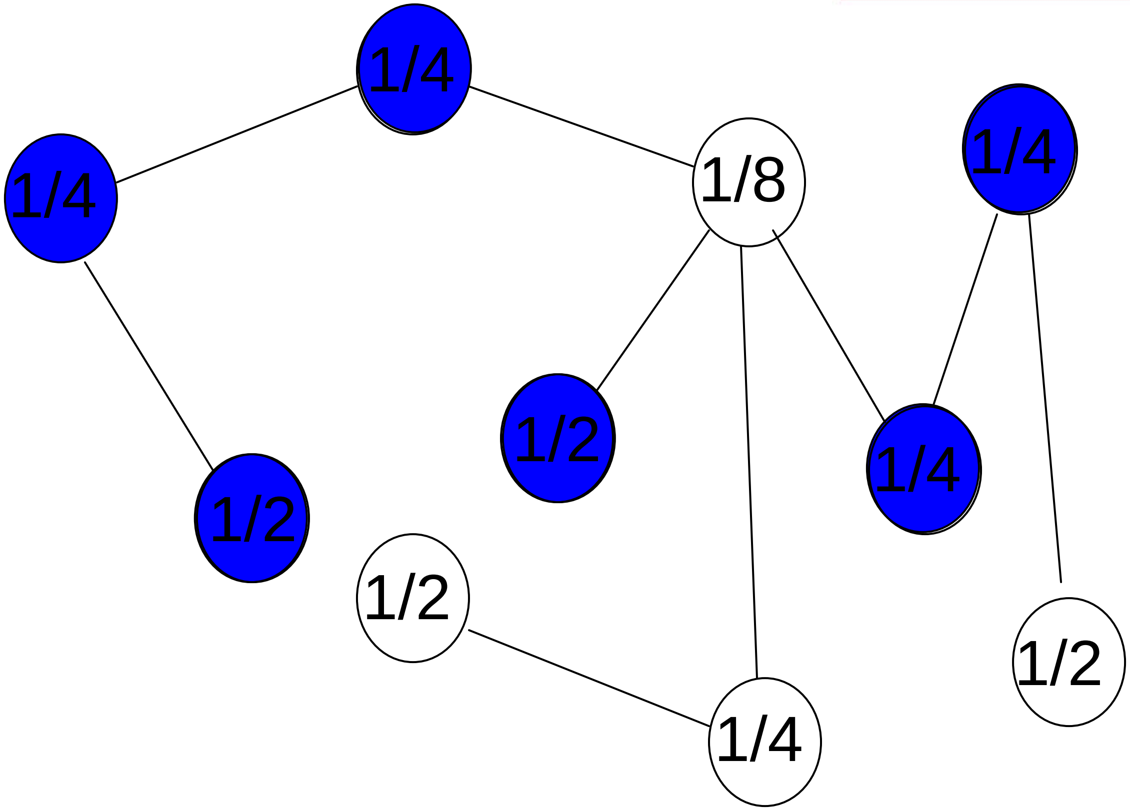
Marking... Who stays?



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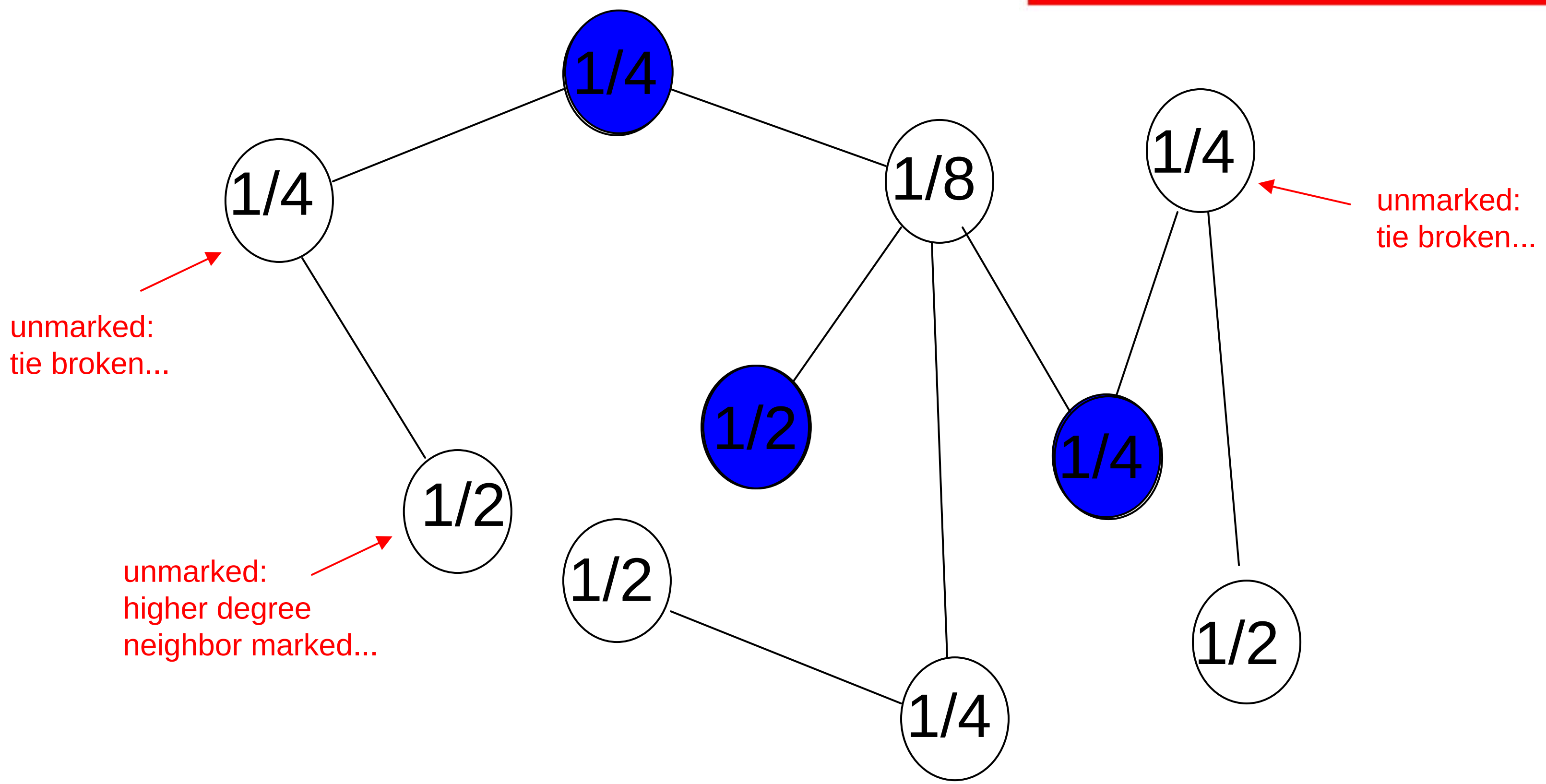
And now?



MIS 1986

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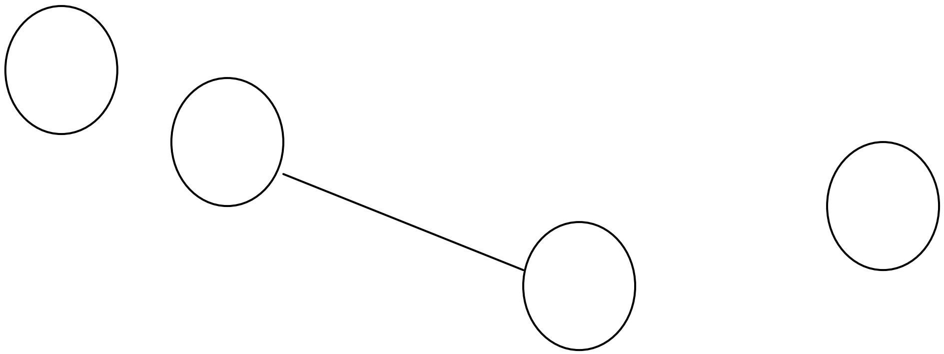
Delete neighborhoods...



Correctness

Fast MIS (1986)

Proceed in rounds consisting of phases

In a phase:

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2. if no higher degree neighbor is marked, v joins MIS; otherwise, v unmarks itself again (break ties arbitrarily)
3. delete all nodes that joined the MIS plus their neighbors, a they cannot join the MIS anymore

IS: Step 1 and Step 2 ensure that node only joins if neighbors do not!

MIS: At some time, nodes will mark themselves in Step 1.

Runtime?

Fast MIS (1986)

Proceed in rounds consisting of phases

In a phase:

1. each node v marks itself with probability 1/2d(v) where d(v) denotes the current degree of v
2. if no higher degree neighbor is marked, v joins MIS; otherwise, v unmarks itself again (break ties arbitrarily)
3. delete all nodes that joined the MIS plus their neighbors, as they cannot join the MIS anymore

Runtime: how fast will algorithm terminate?

Our Strategy!

We want to show logarithmic runtime. So for example?

Idea:

Each node is removed with constant probability (e.g., ½) in each round => half of the nodes vanish in each round.

Or: Each edge is removed with constant probability in each round! As $O(\log m) = O(\log n^2) = O(\log n)$

Unfortunately, this is not true... ☹ Alternative?

A constant fraction of all nodes are removed in each step!
E.g., a constant subset of nodes is „good“ and a constant fraction thereof is removed...

Or the same for edges...

Joining MIS

Node v joins MIS in Step 2 with probability p $\geq$ **?**

Proof.

On what could it depend?

Marked with probability that depends on degree, i.e., 1/2d(v).
(So at most this...)

In MIS subsequently if degree is largest...
(This is likely then if degree is small!)

We will find that marked nodes are likely to join MIS!

Fast MIS (1986)

Proceed in rounds consisting of phases

In a phase:

1. each node v marks itself with probability $1/2d(v)$ where $d(v)$ denotes the current degree of v
2. if no higher degree neighbor is marked, v joins MIS; otherwise, v unmarks itself again (break ties arbitrarily)
3. delete all nodes that joined the MIS plus their neighbors, as they cannot join the MIS anymore

Joining MIS

Node v joins MIS in Step 2 with probability $p \geq 1/(4d(v))$.

Proof.

Let **M** be the set of marked nodes in Step 1.
Let **H(v)** be the set of neighbors of v with higher degree (or same degree and higher identifier).

$$
\begin{aligned}
P[v \notin MIS \mid v \in M] &= P[\exists\, w \in H(v), w \in M \mid v \in M] \\
&= P[\exists\, w \in H(v), w \in M] && \text{// independent whether v is marked or not} \\
&\leq \textstyle\sum_{w \in H(v)} P[w \in M] && \text{// do not only count exactly one but also multiple} \\
&= \textstyle\sum_{w \in H(v)} 1/(2d(w)) && \text{// see Joining MIS algorithm} \\
&\leq \textstyle\sum_{w \in H(v)} 1/(2d(v)) && \text{// v's degree is the lowest one} \\
&\leq d(v)/(2d(v)) = 1/2 && \text{// at most d(v) higher neighbors...}
\end{aligned}
$$

Marked nodes are likely to be in MIS!

So

$$
\begin{aligned}
P[v \in MIS] &= P[v \in MIS \mid v \in M] \cdot P[v \in M] \\
&\geq ½ \cdot 1/(2d(v))
\end{aligned}
$$

QED

Recall Our Strategy!

We want to show logarithmic runtime. So for example?

Idea:

Each node is removed with constant probability (e.g., ½) in each round => half of the nodes vanish in each round.

Or: Each edge is removed with constant probability in each round! As $O(\log m) = O(\log n^2) = O(\log n)$

Unfortunately, this is not true... ☹ Alternative?

Let's try this:

A constant fraction of all nodes are removed in each step!
E.g., a constant subset of nodes is „good" and a constant fraction thereof is removed...

How to define good nodes?!
Node with low degree neighbors!
(Why? Likely to be removed as neighbors are likely to be marked and hence join MIS...)

Or the same for edges...

Fast MIS (1986)

Proceed in rounds consisting of phases

In a phase:

1. each node v marks itself with probability 1/2d(v) where d(v) denotes the current degree of v
2. if no higher degree neighbor is marked, v joins MIS; otherwise, v unmarks itself again (break ties arbitrarily)
3. delete all nodes that joined the MIS plus their neighbors, as they cannot join the MIS anymore

Good&Bad Nodes

A node v is called *good* if

$$\sum_{w \in N(v)} 1/(2d(w)) \geq 1/6.$$

What does it mean?

A good node has neighbors of low degree. Likely to be removed when neighbor joins MIS!

Good Nodes

A good node v will be removed in Step 3 with probability

$$p \geq 1/36.$$

Proof?

Analysis (1)

„Assets“:

Joining MIS

Node v joins MIS in Step 2 with probability $p \geq 1/(4d(v))$.

Good&Bad Nodes

A node v is called *good* if

$\Sigma_{w \in N(v)} 1/(2d(w)) \geq 1/6$.

Proof („Good Nodes“).

Goal:

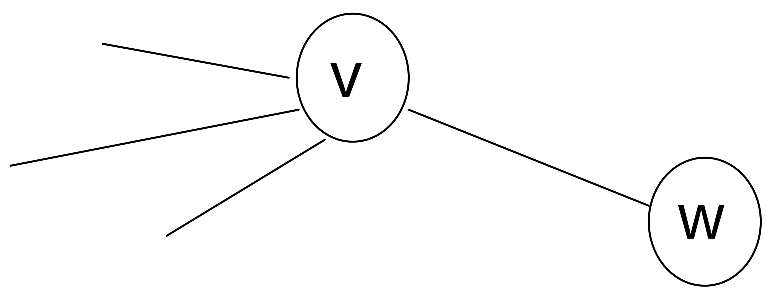
Good Nodes

A good node v will be removed in Step 3 with probability

$p \geq 1/36$.

If v has a neighbor w with $d(w) \leq 2$?

Done: „Joining MIS“ lemma implies that prob. to remove at least 1/8 since neighbor w will join...



So let‘s focus on neighbors with degree at least 3: thus for any neighbor w of v we have $1/(2d(w)) \leq 1/6$.

Analysis (2)

„Assets“:

Joining MIS

Node v joins MIS in Step 2 with probability $p \geq 1/(4d(v))$.

Good&Bad Nodes

A node v is called *good* if

$\Sigma_{w \in N(v)} 1/(2d(w)) \geq 1/6$.

Proof („Good Nodes“).

Goal:

Good Nodes

A good node v will be removed in Step 3 with probability

$p \geq 1/36$.

So neighbors have degree at least 3...

Then, for a good node v, there must be a subset $S \subseteq N(v)$ such that

$$1/6 \leq \Sigma_{w \in S} 1/(2d(w)) \leq 1/3.$$

Why?

By taking all neighbors we have at least 1/6 (Definition), and we can remove individual nodes with a granularity of at least 1/6 (degree at least 3).

Analysis (3)

Good Nodes

A good node v will be removed in Step 3 with probability

$p \geq 1/36$.

Proof („Good Nodes“).

Let **R** be event that v is removed (e.g., if neighbor joins MIS).

$P[R] \geq P[\exists\, u \in S, u \in MIS]$ // removed e.g., if neighbor joins

$\geq \sum_{u \in S} P[u \in MIS] - \sum_{u,w \in S} P[u \in MIS \text{ and } w \in MIS]$ // why?

By truncating the inclusion-exclusion priniple...:
Probability that there is one is sum of probability for all individual minus probability that two enter, plus...

just derived!

$$P[R] \geq \sum_{u \in S} P[u \in \text{MIS}] - \sum_{u,w \in S; u \neq w} P[u \in M \ \textbf{and}\ w \in M]$$

independent but count same node double in sum...

$$\geq \sum_{u \in S} P[u \in \text{MIS}] - \sum_{u \in S} \sum_{w \in S} P[u \in M] \cdot P[w \in M]$$

see Joining MIS lemma

see algorithm

$$\geq \sum_{u \in S} \frac{1}{4d(u)} - \sum_{u \in S} \sum_{w \in S} \frac{1}{2d(u)} \frac{1}{2d(w)}$$

$$\geq \sum_{u \in S} \frac{1}{2d(u)} \left(\frac{1}{2} - \sum_{w \in S} \frac{1}{2d(w)} \right) \geq \frac{1}{6} \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{1}{36}.$$

QED

Analysis

We just proved:

Good Nodes

A good node v will be removed in Step 3 with probability

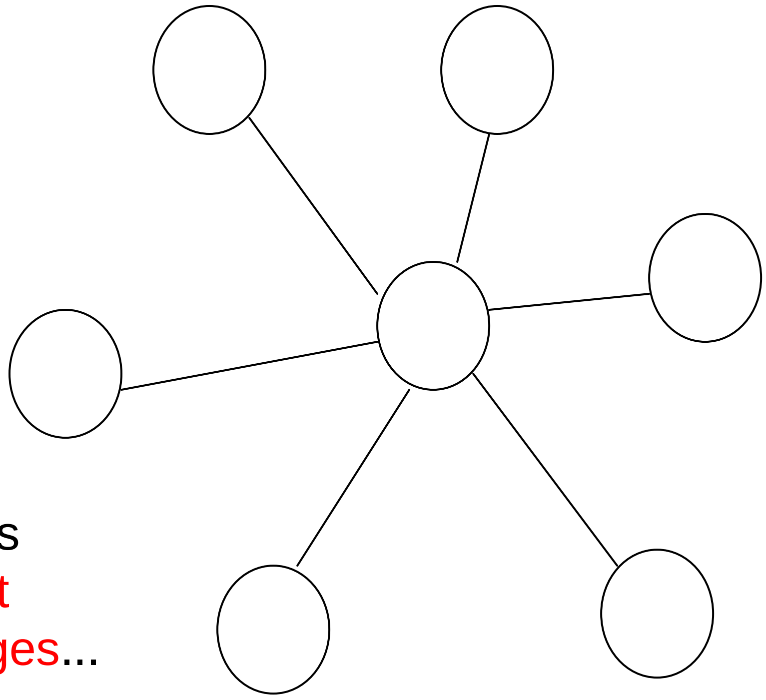
$$p \geq 1/36.$$

Cool, good nodes have
constant probability! ☺
But what now?
What does it help?
Are many nodes good in a graph?

Example: in star graph,
only single node is good... ☹

But: there are many „good edges“...
How to define good edges?

Idea: edge is removed if either of its endpoints are removed! So good if at least one endpoint is a good node! And there are many such edges...



Analysis

Fast MIS (1986)

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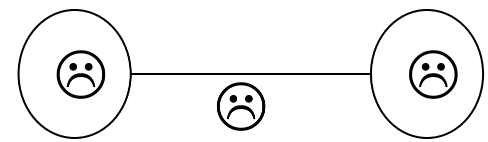
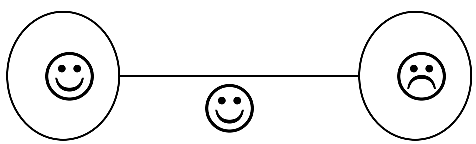
3. delete all nodes that joined the MIS plus their neighbors, as they cannot join the MIS anymore

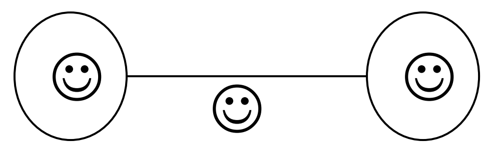
Good&Bad Edges

An edge e=(u,v) called *bad* if both u and v are bad (not good). Else the edge is called good.

A bad edge is incident to two nodes with neighbors of high degrees.





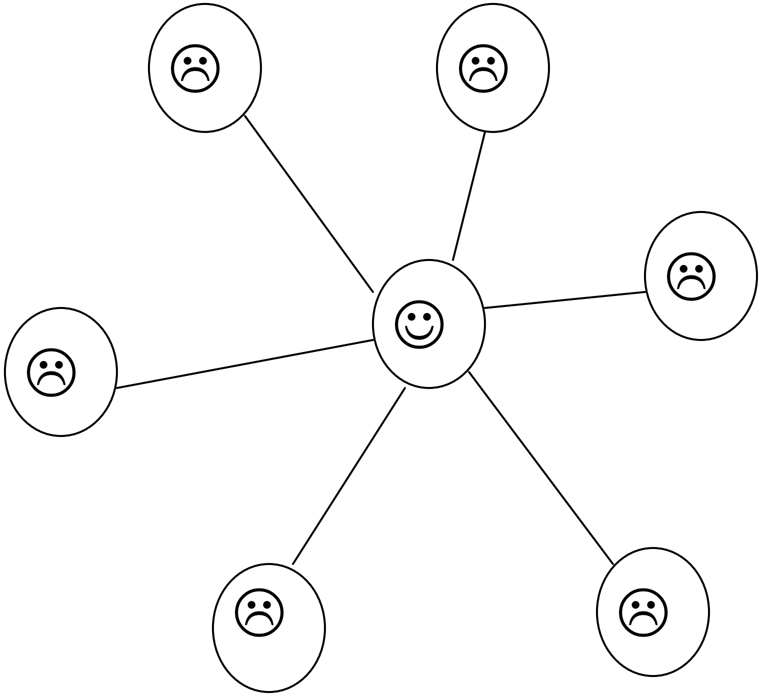


Good Edges

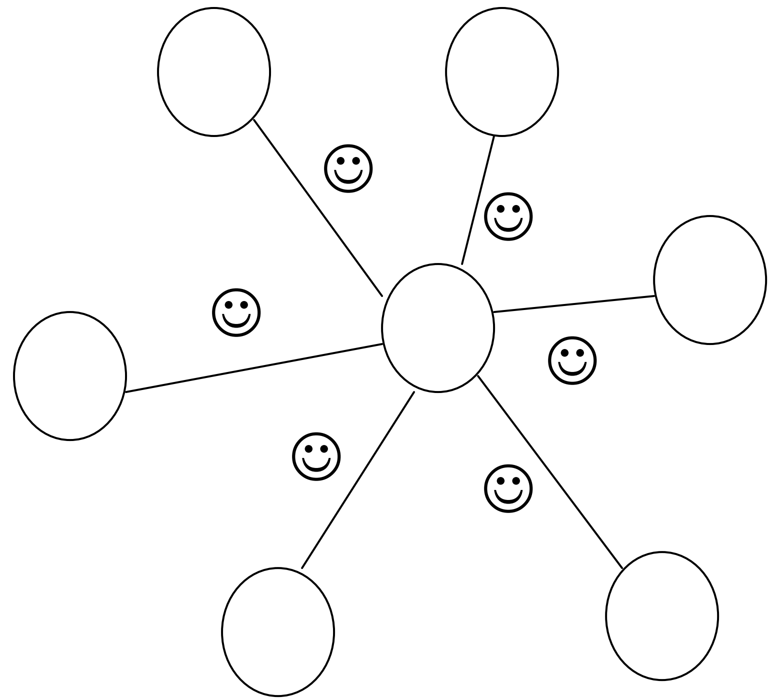
At least half of all edges are good, at any time.

Proof?

Analysis



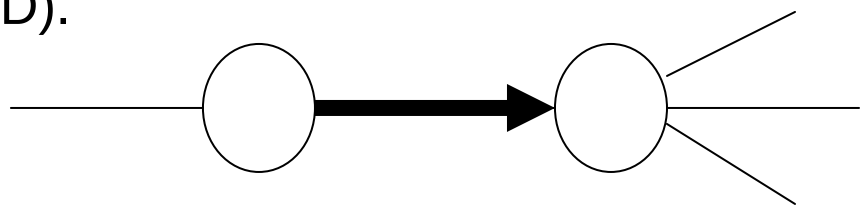
Not many good nodes...

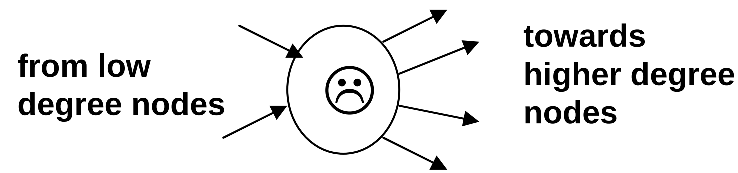


... but many good edges!

Analysis

Idea: Construct an auxiliary graph! Direct each edge towards higher degree node (if both nodes have same degree, point it to one with higher ID).





Helper Lemma

A bad node v has out-degree at least twice its indegree.

Proof („Helper Lemma“).

Assume the opposite: at least $d(v)/3$ neighbors (let‘s call them $S \subseteq N(v)$) have degree at most $d(v)$ (otherwise v would point to them). But then

$$\sum_{w \in N(v)} \frac{1}{2d(w)} \geq \sum_{w \in S} \frac{1}{2d(w)} \geq \sum_{w \in S} \frac{1}{2d(v)} \geq \frac{d(v)}{3} \frac{1}{2d(v)} = \frac{1}{6}$$

only subset...

Def. of S

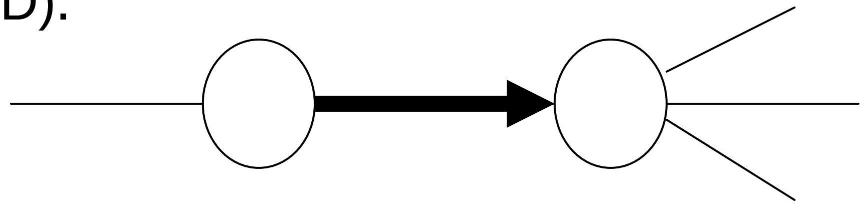
Assumption

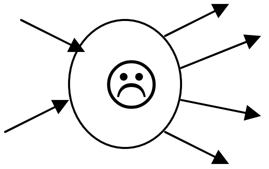
Contradiction: v would be good!

QED

Analysis

Idea: Construct an auxiliary graph! Direct each edge towards higher degree node (if both nodes have same degree, point it to one with higher ID).





Helper Lemma

A bad node v has out-degree at least twice its indegree.

So what?

The number of edges into bad nodes can be at most half the number of all edges!
So at least half of all edges are directed into good nodes!
And they are good! ☺

Fast MIS (1986)

Fast MIS terminates in expected time O(log n).

Proof („Fast MIS“)?

We know that a good node will be deleted with constant probability in Step 3 (but there may not be many). And with it, a good edge (by definition)!

Since at least half of all the edges are good (and thus have at least one good incident node which will be deleted with constant probability and so will the edge!), a constant fraction of edges will be deleted in each phase.
(Note that O(log m)=O(log n).)

QED

Back to the future:
Fast MIS from 2009…!

Even simpler algorithm!

Fast MIS from 2009...

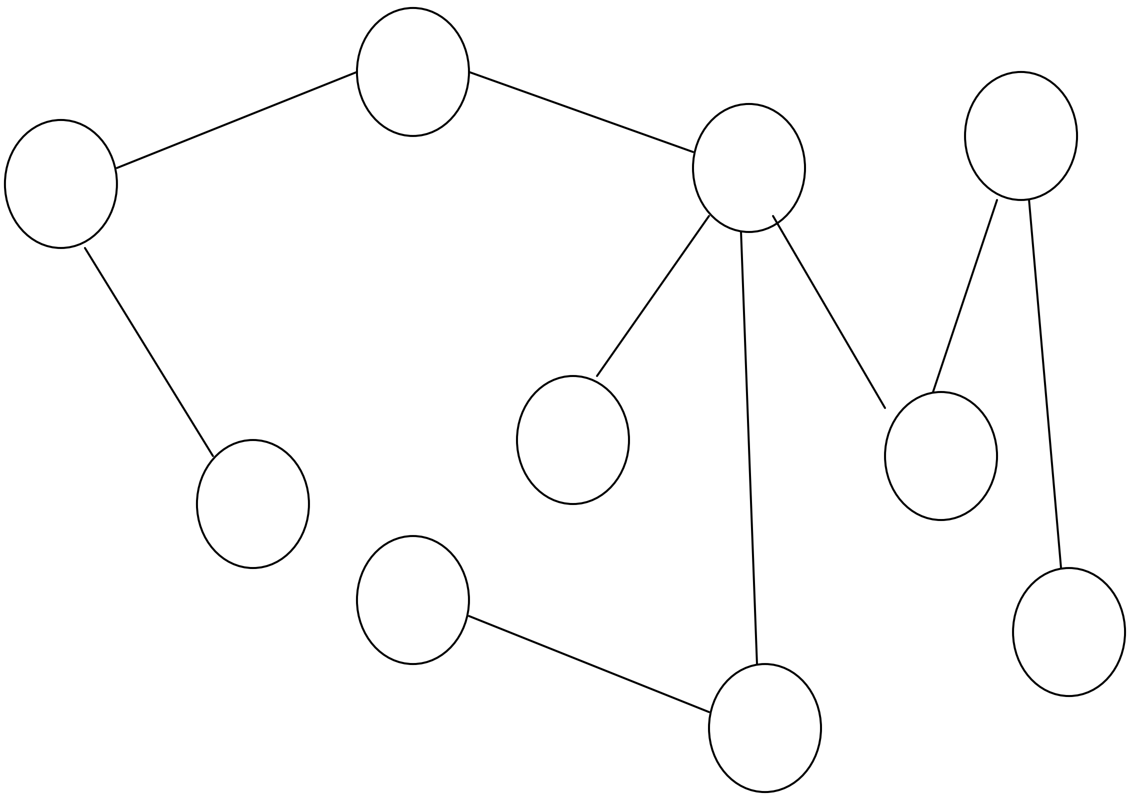
Fast MIS (2009)

Proceed in rounds consisting of phases!

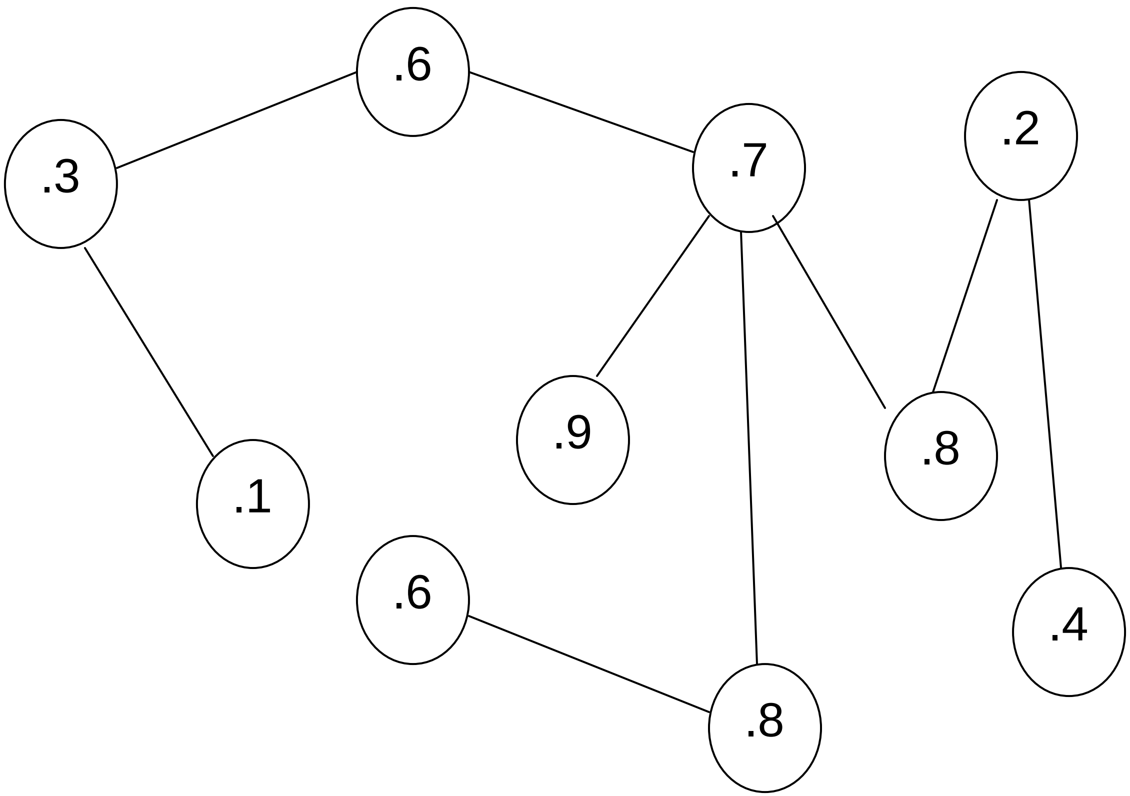
In a phase:

1. each node chooses a random value **r(v)** $\in$ [0,1] and sends it to ist neighbors.
2. If **r(v) < r(w)** for all neighbors w $\in$ N(v), node v enters the MIS and informs the neighbors
3. If v or a neighbor of v entered the MIS, v **terminates** (and v and edges are **removed**), otherwise v enters next phase!

Fast MIS from 2009...

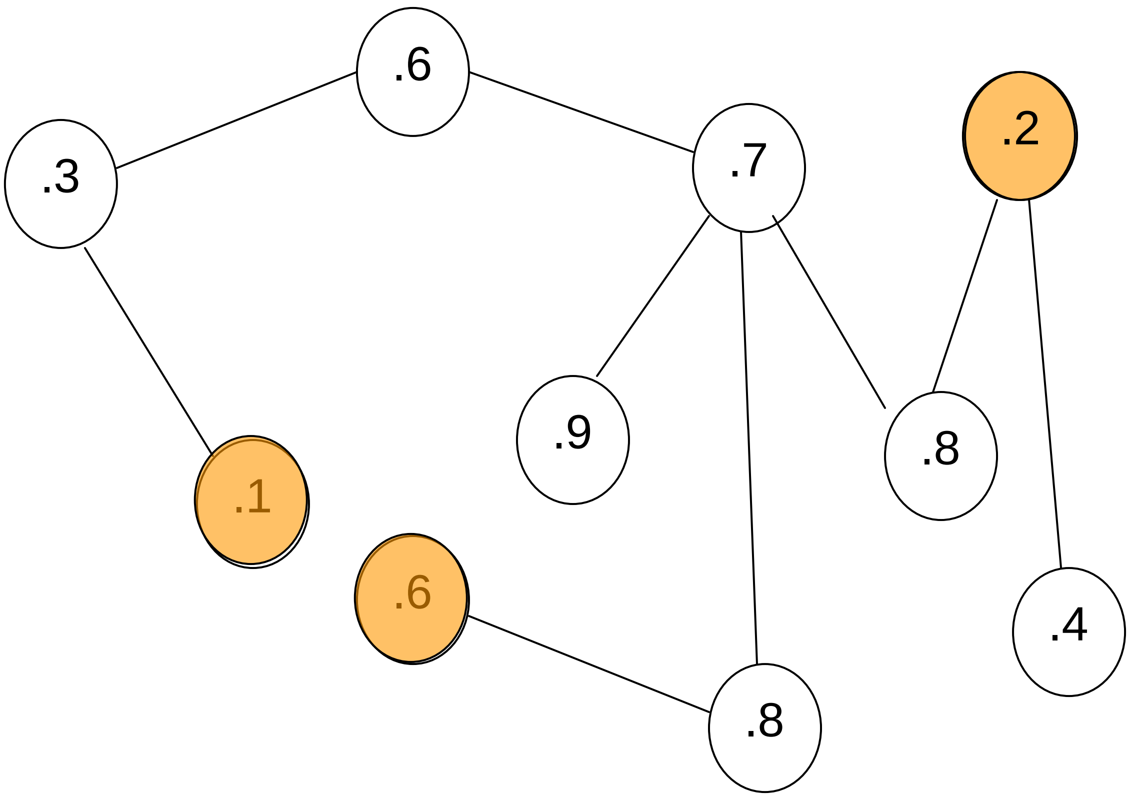


Fast MIS from 2009...



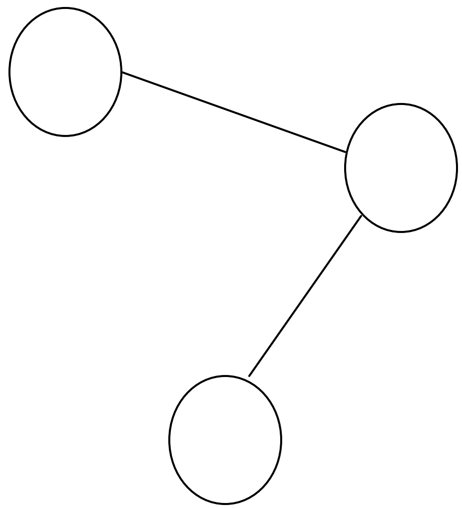
Choose random values!

Fast MIS from 2009...



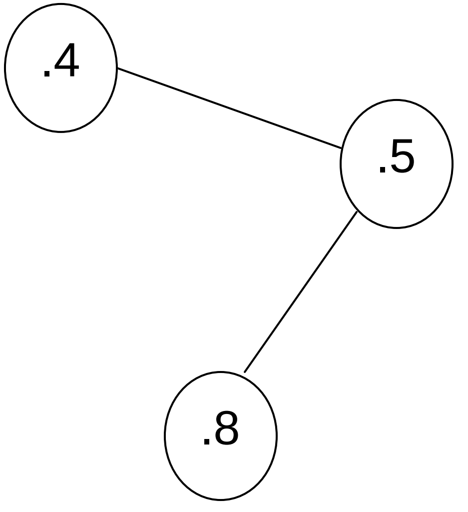
Min in neighborhood => IS!

Fast MIS from 2009...



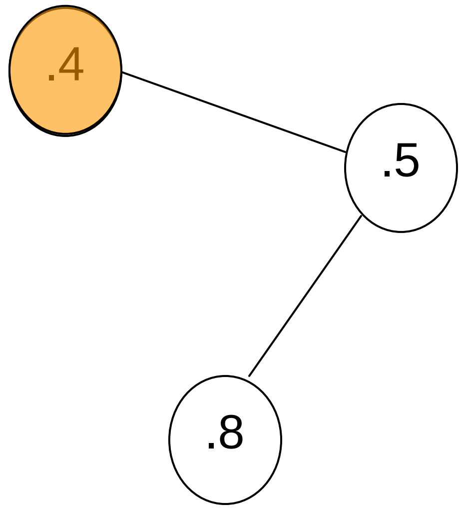
Remove neighborhoods...

Fast MIS from 2009...



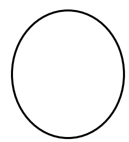
Choose random values!

Fast MIS from 2009...



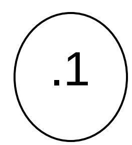
Min in neighborhood => IS!

Fast MIS from 2009...



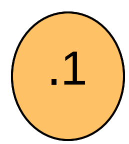
Remove neighborhoods...

Fast MIS from 2009...



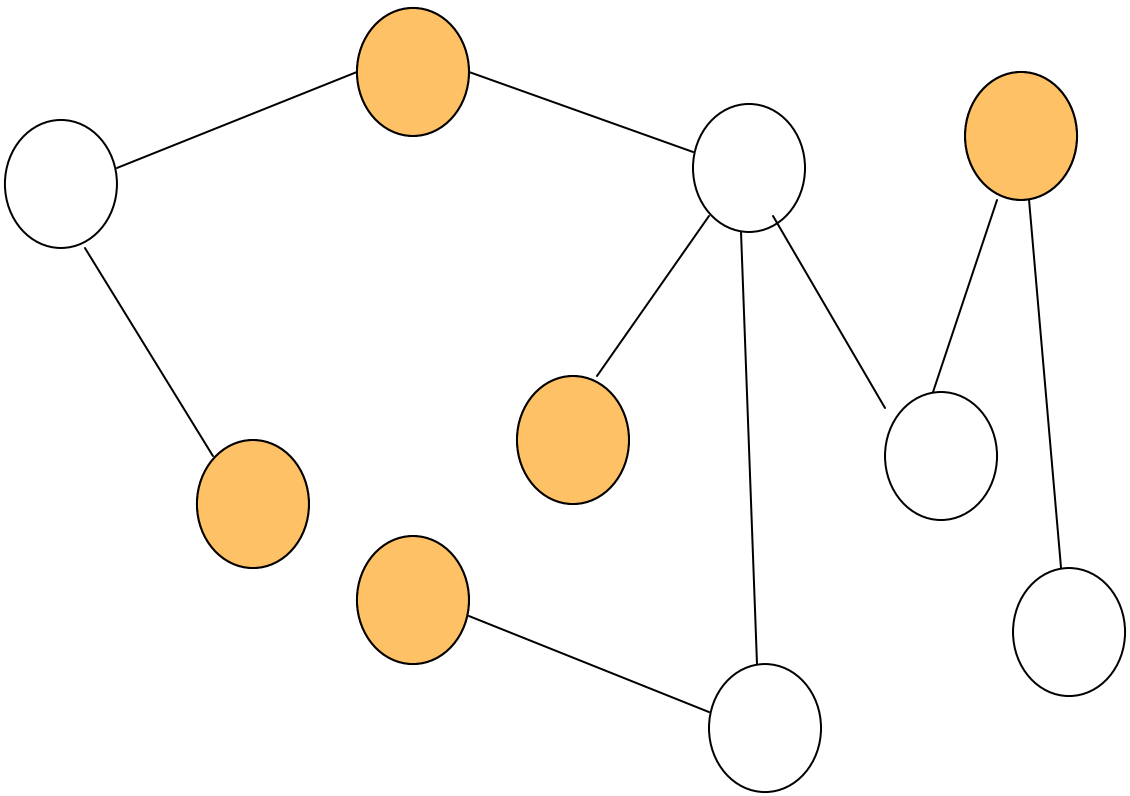
Choose random values!

Fast MIS from 2009...



lowest value => IS

Fast MIS from 2009...



... done: MIS!

Fast MIS from 2009...

Fast MIS (2009)

Proceed in rounds consisting of phases!

In a phase:

1. each node chooses a random value **r(v)** $\in$ [0,1] and sends it to ist neighbors.
2. If **r(v) < r(w)** for all neighbors w $\in$ N(v), node v enters the MIS and informs the neighbors
3. If v or a neighbor of v entered the MIS, v **terminates** (and v and edges are **removed**), otherwise v enters next phase!

Why is it correct? Why IS?

Step 2: if v joins, neighbors do not

Step 3: if v joins, neighbors will *never* join again

Fast MIS from 2009...

Fast MIS (2009)

Proceed in rounds consisting of phases!

In a phase:

1. each node chooses a random value **r(v)** $\in$ [0,1] and sends it to ist neighbors.
2. If **r(v) < r(w)** for all neighbors w $\in$ N(v), node v enters the MIS and informs the neighbors
3. If v or a neighbor of v entered the MIS, v **terminates** (and v and edges are **removed**), otherwise v enters next phase!

Why MIS?

Node with smalles random value will always join the IS, so there is always progress.

Fast MIS from 2009...

Fast MIS (2009)

Proceed in rounds consisting of phases!

In a phase:

1. each node chooses a random value **r(v)** $\in$ [0,1] and sends it to ist neighbors.
2. If **r(v) < r(w)** for all neighbors w $\in$ N(v), node v enters the MIS and informs the neighbors
3. If v or a neighbor of v entered the MIS, v **terminates** (and v and edges are **removed**), otherwise v enters next phase!

Runtime?

Analysis: Recall „Linearity of Expectation“

Theorem 5.9 (Linearity of Expectation). *Let X_i, $i = 1, \ldots, k$ denote random variables, then*

$$\mathbb{E}\left[\sum_i X_i\right] = \sum_i \mathbb{E}\left[X_i\right].$$

Proof. It is sufficient to prove $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ for two random variables X and Y, because then the statement follows by induction. Since

$$\begin{aligned} P[(X,Y) = (x,y)] &= P[X = x] \cdot P[Y = y | X = x] \\ &= P[Y = y] \cdot P[X = x | Y = y] \end{aligned}$$

we get that

$$\begin{aligned} \mathbb{E}[X+Y] &= \sum_{(X,Y)=(x,y)} P[(X,Y) = (x,y)] \cdot (x+y) \\ &= \sum_{X=x} \sum_{Y=y} P[X = x] \cdot P[Y = y | X = x] \cdot x \\ &+ \sum_{Y=y} \sum_{X=x} P[Y = y] \cdot P[X = x | Y = y] \cdot y \\ &= \sum_{X=x} P[X = x] \cdot x + \sum_{Y=y} P[Y = y] \cdot y \\ &= \mathbb{E}[X] + \mathbb{E}[Y]. \end{aligned}$$

We sum over all possible y values for a given x, so =1

Analysis? (1)

We want to show that also this algorithm
has logarithmic runtime! How?

Idea: if per phase a constant fraction of node disappeared,
it would hold! (Recall definition of logarithm...)

Again: this is not true unfortunately... ☹

Alternative proof? Similar to last time?

Show that any *edge* disappears with constant probability!

But also this does not work: edge does not have
constant probability to be removed!
But maybe edges still vanish quickly...?

Let's estimate the number of disappearing edges per round again!

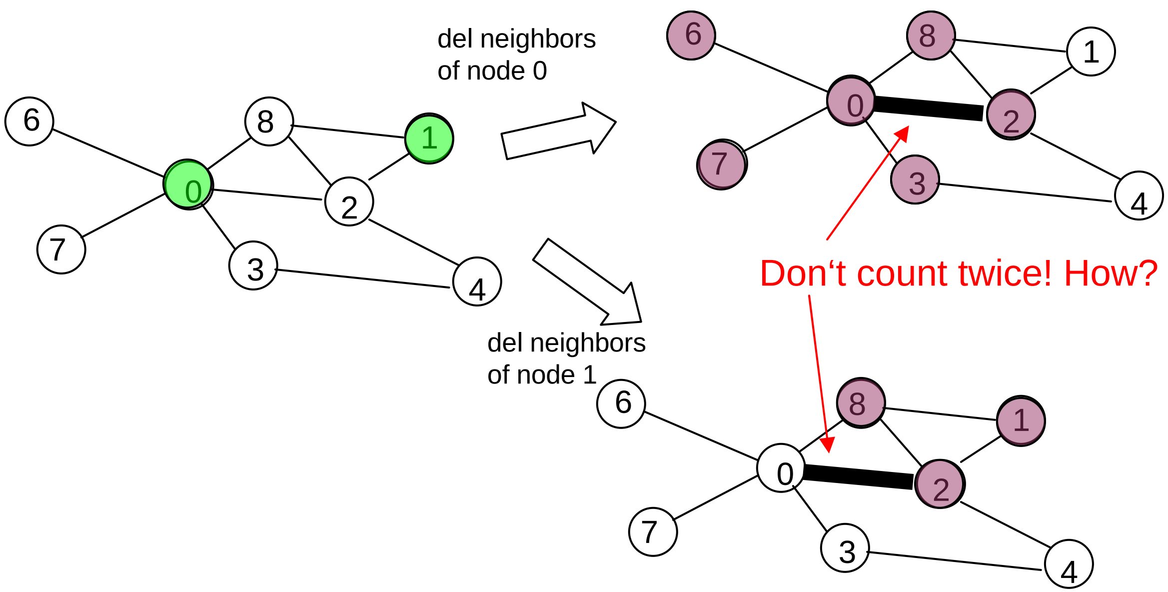
Analysis? (2)

Probability of a node v to enter MIS?

Probability = node v has largest ID in neighborhood, so at least 1/(d(v)+1)...

... also v‘s neighbors‘ edges will disappear with this probability, so more than d(v) edges go away with this probability!

But let‘s make sure we do not double count edges!



Idea: only count edges from a neighbor w when v is the smallest value even in w‘s neighborhood!
It‘s a subset only, but sufficient!

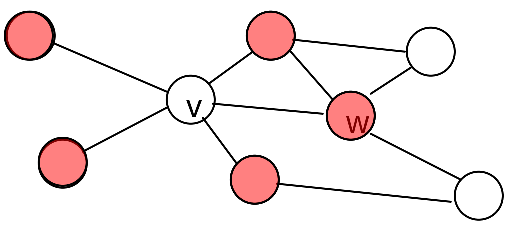
Edge Removal: Analysis (1)

Edge Removal

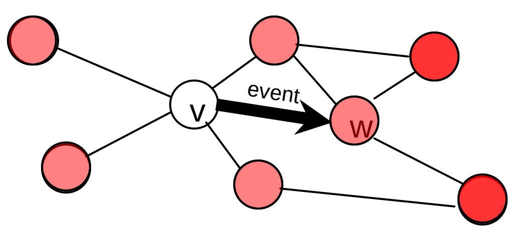
In expectation, we remove at least half of all the edges in any phase.

Proof („Edge Removal“)?

Consider the graph G=(V,E), and assume **v joins MIS** (i.e., r(v)<r(w) for all neighbors w).
If in addition, it holds that r(v)<r(x) for all neighbors x of a neighbor w, we call this **event (v => w)**.

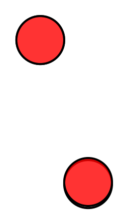


What is the probability of this event (that v is minimum also for neighbors of the given neighbor)?



$$P[(v => w)] \geq 1/(d(v)+d(w)),$$

since d(v)+d(w) is the maximum possible number of nodes adjacent to v and w.
If v joins MIS, all edges (w,x) will be removed; there are at least d(w) many.



Edge Removal: Analysis (2)

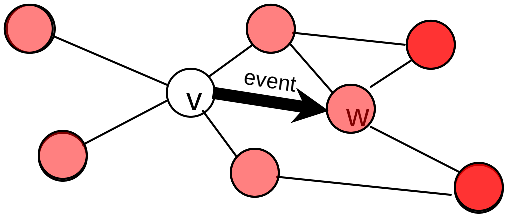
Edge Removal

In expectation, we remove at least half of all the edges in any phase.

Proof („Edge Removal")?

How many edges are removed?
Let $X_{(v=>w)}$ denote random variable for number of edges adjacent to w removed due to event (v=>w). If (v=>w) occurs, $X_{(v=>w)}$ has value d(w), otherwise 0.
Let **X** denote the sum of all these random variables.
So:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{\{v,w\}\in E} \mathbb{E}[X_{(v\to w)}] + \mathbb{E}[X_{(w\to v)}] \\
&= \sum_{\{v,w\}\in E} P[\text{Event } (v\to w)]\cdot d(w) + P[\text{Event } (w\to v)]\cdot d(v) \\
&\geq \sum_{\{v,w\}\in E} \frac{d(w)}{d(v)+d(w)} + \frac{d(v)}{d(w)+d(v)} \\
&= \sum_{\{v,w\}\in E} 1 = |E|.
\end{aligned}$$



So all edges gone in one phase?!
We still overcount!

Edge Removal: Analysis (3)

Edge Removal

In expectation, we remove at least half of all the edges in any phase.

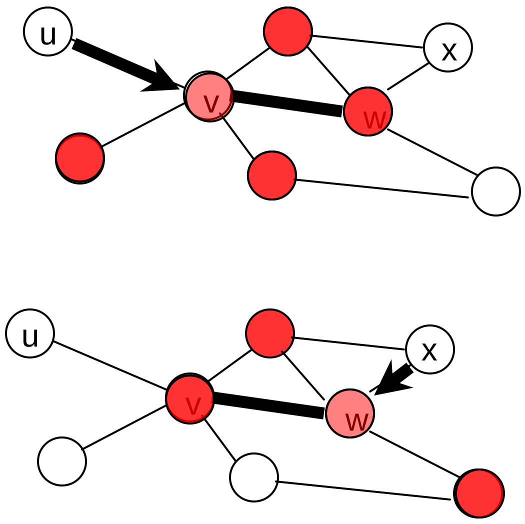
Proof („Edge Removal")?

We still overcount:
Edge {v,w} may be counted twice:
for event (u=>v) and event (x=>w).

However, it cannot be more than twice, as there is at most one event (*=>v) and at most one event (*=>w):

Event (u=>v) means $r(u)<r(w)$ for all $w \in N(v)$; another (u'=>v) would imply that $r(u')>r(u) \in N(v)$.

So at least half of all edges vanish!



QED

2009 MIS: Analysis

MIS of 2009

Expected running time is O(log n).

Proof („MIS 2009“)?

Number of edges is cut in two in each round...

QED

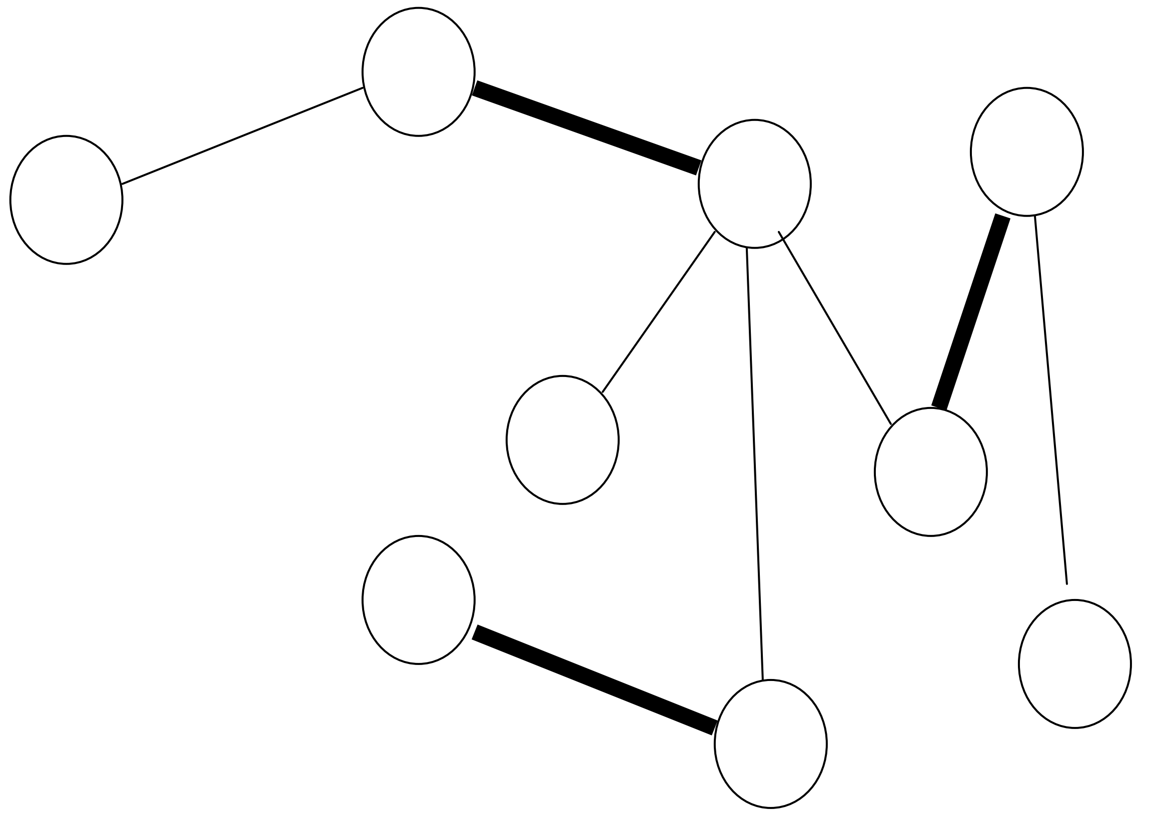
Actually, the claim even holds with high probability! (see „Skript“)

Excursion: Matchings

Matching

A matching is a subset M of edges E such that no two edges in M are adjacent.
A maximal matching cannot be augmented.
A maximum matching is the best possible.
A perfect matching includes all nodes.

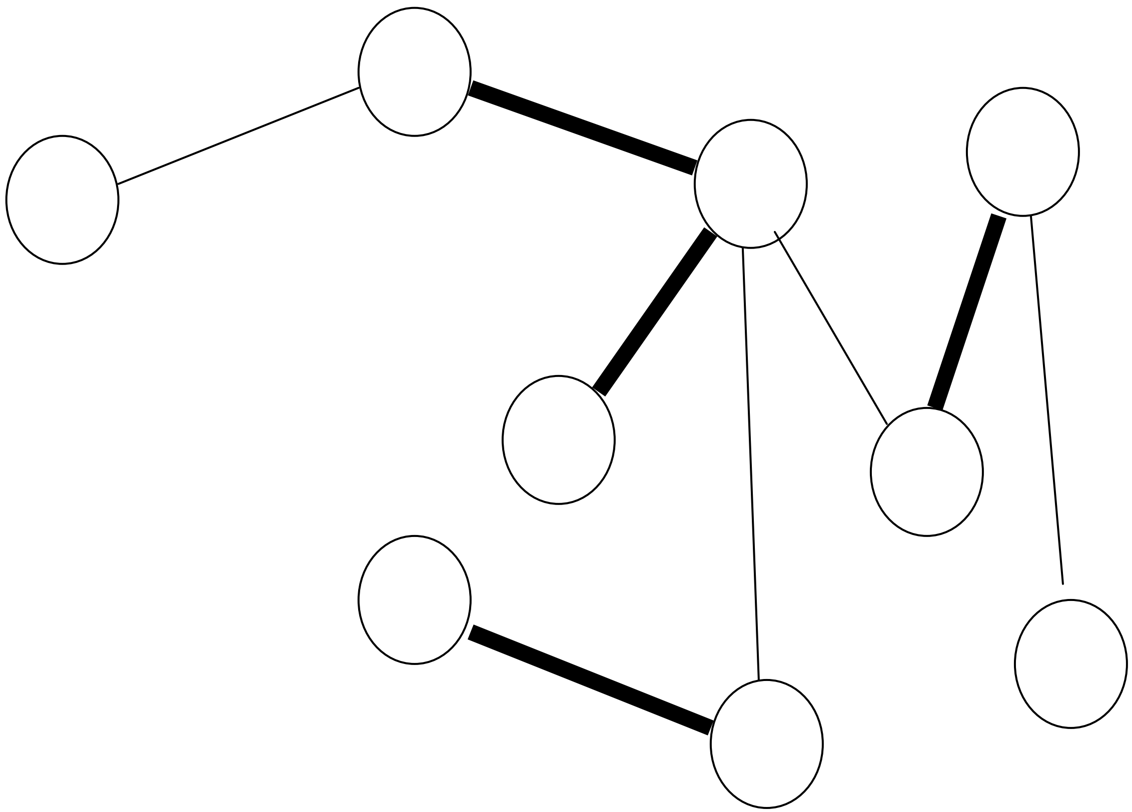
Excursion: Matchings



Matching? Maximal? Maximum? Perfect?

Maximal.

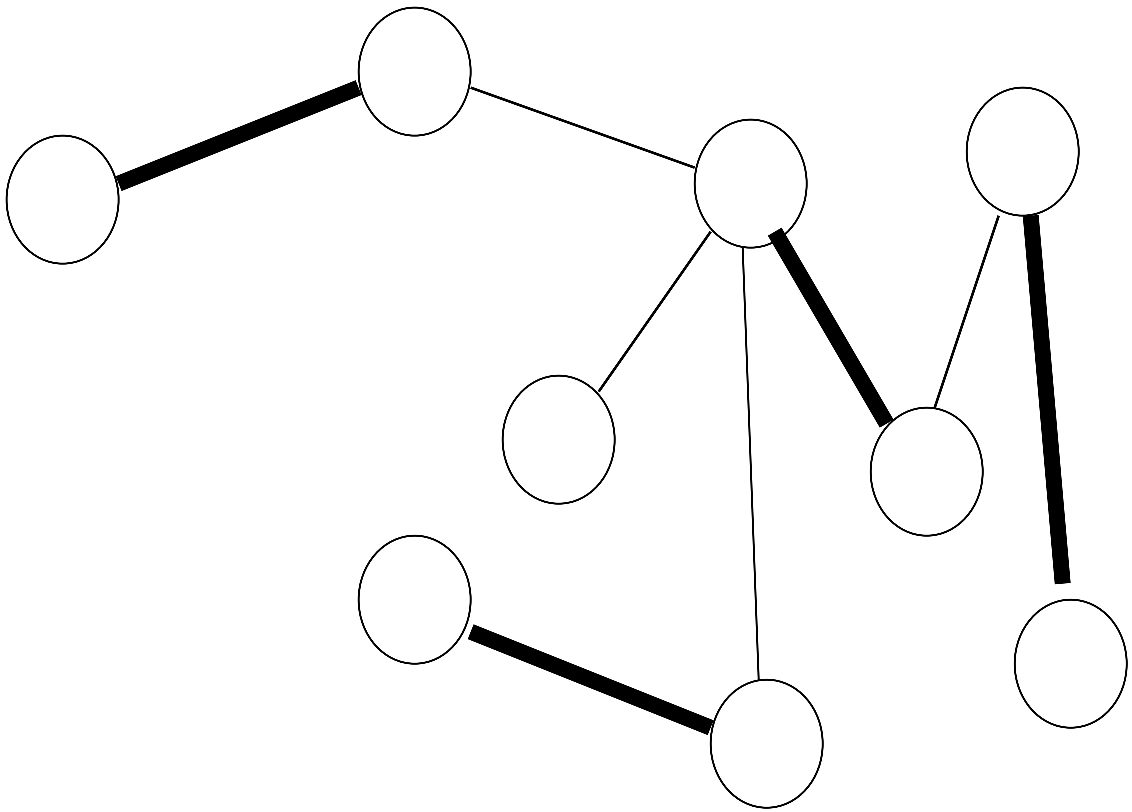
Excursion: Matchings



Matching? Maximal? Maximum? Perfect?

Nothing.

Excursion: Matchings



Matching? Maximal? Maximum? Perfect?

Maximum but not perfect.

Discussion: Matching

Matching

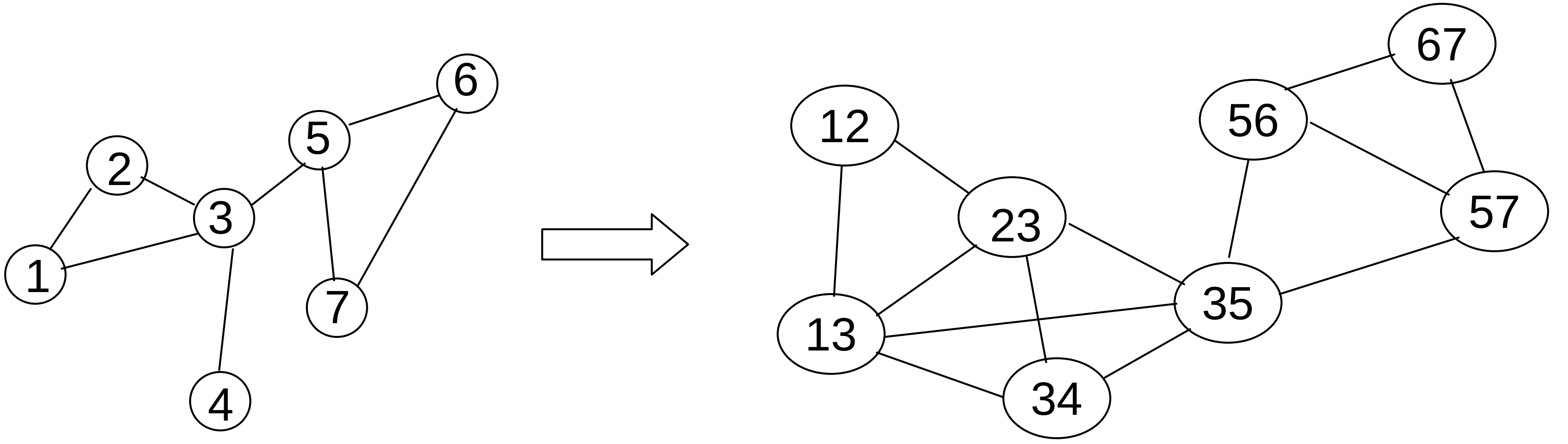
A matching is a subset M of edges E such that no two edges in M are adjacent.
A maximal matching cannot be augmented.
A maximum matching is the best possible.
A perfect matching includes all nodes.

How to compute with an IS algorithm?

Discussion: Matching

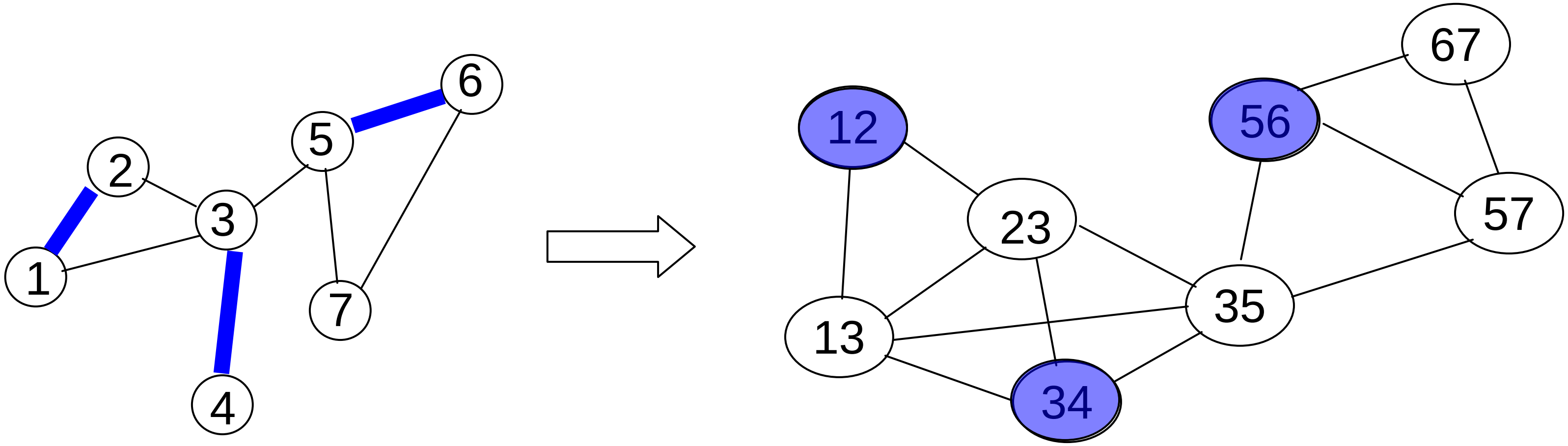
An IS algorithm is a matching algorithm! How?

For each edge in original graph make vertex, connect vertices if their edges are adjacent.



Discussion: Matching

MIS = maximal matching: matching does not have adjacent edges!



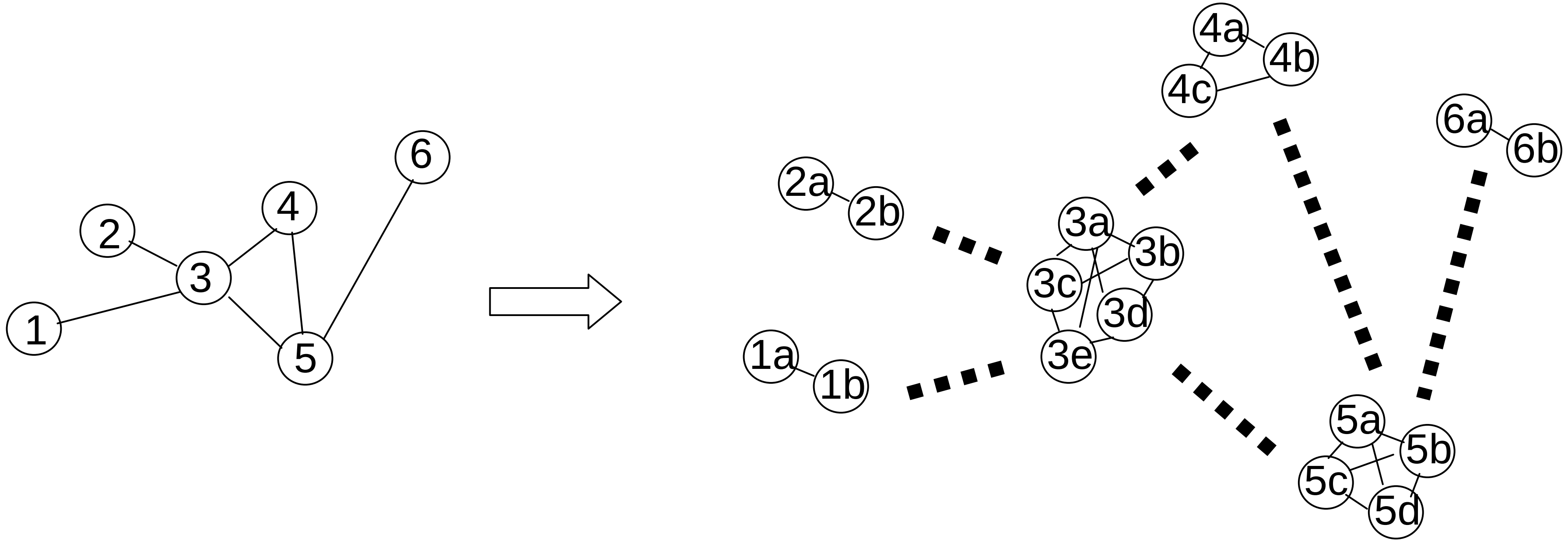
Discussion: Graph Coloring

How to use a MIS algorithm for graph coloring?

How to use a MIS algorithm for graph coloring?

Clone each node v, d(v)+1 many times. Connect clones completely and edges from i-th clone to i-th clone. Then?

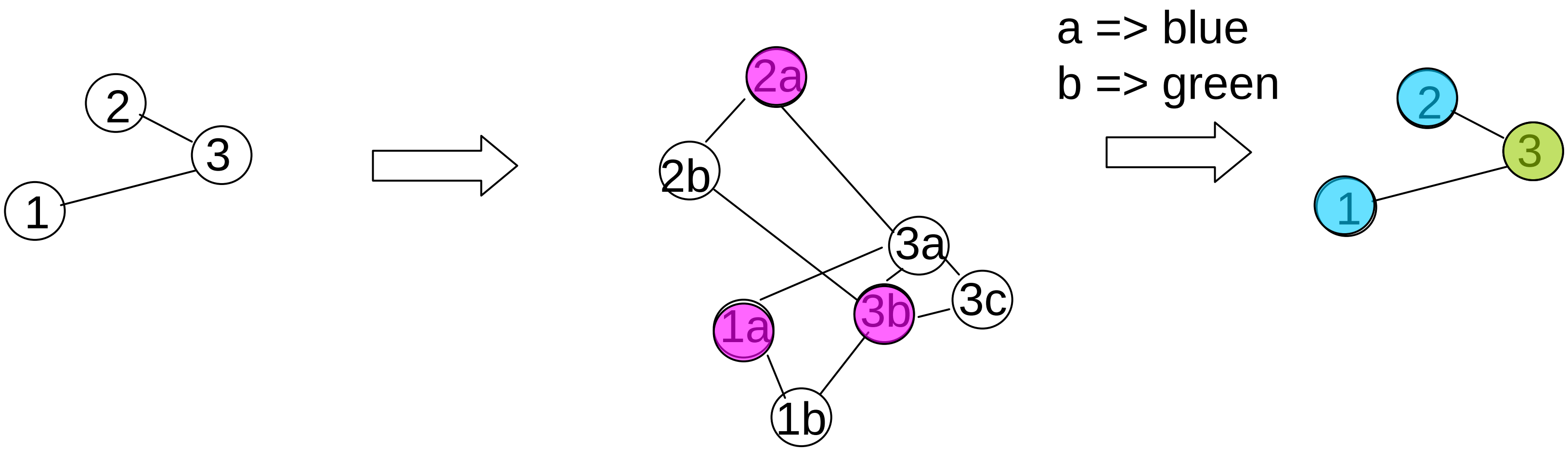
Run MIS: if i-th copy is in MIS, node gets color i.



Discussion: Graph Coloring

Example:

How to use a MIS algorithm for graph coloring?

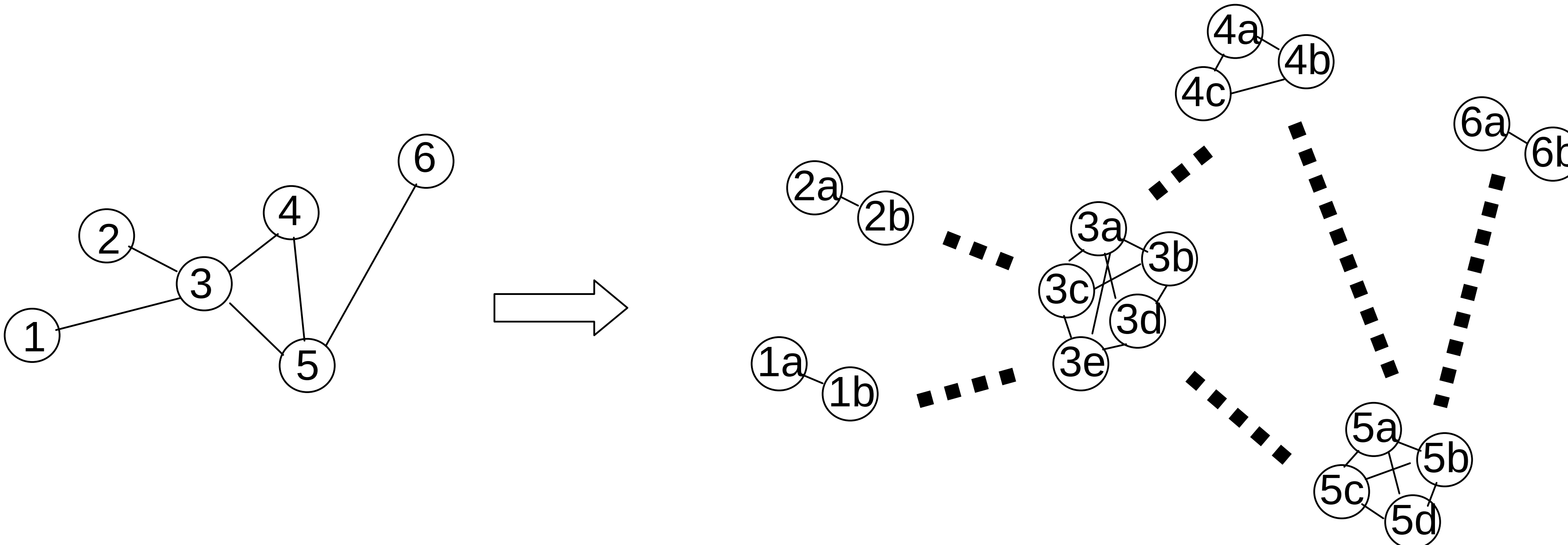


MIS

Coloring

Discussion: Graph Coloring

Why does it work?



1. Idea conflict-free: adjacent nodes cannot get same color (different index in MIS, otherwise adjacent!), and each node has at most one clone in IS, so valid.
2. Idea colored: each node gets color, i.e., each node has a clone in IS: there are only d(v) neighbor clusters, but our cluster has d(v)+1 nodes...

Discussion: Dominating Set

Dominating Set

A subset D of nodes such that each node either is in the dominating set itself, or one of ist neighbors is (or both).

How to compute a dominating set?
See Skript. ☺

Literature for further reading:

- Peleg‘s book (as always ☺)

End of lecture