Overview

• Mobile IP
  – Motivation
  – Data transfer
  – Encapsulation

• Location Services & Routing
  – Classification of location services
  – Home based
  – GLS
  – MLS
  – Example: Location service of GSM

• Mobility Models

Motivation for Mobile IP

• Routing
  – based on IP destination address, network prefix (e.g. 129.132.13) determines physical subnet
  – change of physical subnet implies change of IP address to have a topological correct address (standard IP) or needs special entries in the routing tables

• Changing the IP-address?
  – adjust the host IP address depending on the current location
  – almost impossible to find a mobile system, DNS updates are too slow
  – TCP connections break
  – security problems

• Change/Add routing table entries for mobile hosts?
  – worldwide!
  – does not scale with the number of mobile hosts and frequent changes in their location

Rating

• Area maturity
  First steps
  Text book

• Practical importance
  No apps
  Mission critical

• Theoretical importance
  Not really
  Must have
Requirements on Mobile IP (RFC 2002)

- **Compatibility**
  - support of the same layer 2 protocols as IP
  - no changes to current end-systems and routers required
  - mobile end-systems can communicate with fixed systems

- **Transparency**
  - mobile end-systems keep their IP address
  - continuation of communication after interruption of link possible
  - point of connection to the fixed network can be changed

- **Efficiency and scalability**
  - only little additional messages to the mobile system required
    (connection typically via a low bandwidth radio link)
  - world-wide support of a large number of mobile systems

- **Security**
  - authentication of all registration messages

Data transfer from mobile system:
1. Sender sends to the IP of MN,
   HA intercepts packet (proxy ARP)
2. HA tunnels packet to FA by encapsulation
3. FA forwards packet to the MN

Data transfer to mobile system:
1. MN sends to the IP address of the receiver (CN) as usual, FA works as default router.
Terminology

- **Mobile Node (MN)**
  - system (node) that can change the point of connection to the network without changing its IP address
- **Home Agent (HA)**
  - system in the home network of the MN, typically a router
  - registers the location of the MN, tunnels IP datagrams to the COA
- **Foreign Agent (FA)**
  - system in the current foreign network of the MN, typically a router
    - typically the default router for the MN
- **Care-of Address (COA)**
  - address of the current tunnel end-point for the MN (at FA or MN)
  - actual location of the MN from an IP point of view
    - can be chosen, e.g., via DHCP
- **Correspondent Node (CN)**

Overview

**Agent Advertisement**
- HA and FA periodically send advertisement messages into their physical subnets
- MN listens to these messages and detects if it is in the home or a foreign network (standard case for home network)
- MN reads a COA from the FA advertisement messages

**Registration (always limited lifetime!)**
- MN signals COA to the HA via the FA, HA acknowledges via FA to MN
  - these actions have to be secured by authentication

**Advertisement**
- HA advertises the IP address of the MN (as for fixed systems), i.e.
  - standard routing information
- routers adjust their entries, these are stable for a longer time (HA responsible for a MN over a longer period of time)
- packets to the MN are sent to the HA
- independent of changes in COA/FA
IP-in-IP Encapsulation

- Mandatory in RFC 2003
- tunnel between HA and COA

<table>
<thead>
<tr>
<th>ver.</th>
<th>IHL</th>
<th>TOS</th>
<th>length</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IP identification</th>
<th>flags</th>
<th>fragment offset</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TTL</th>
<th>IP-in-IP</th>
<th>IP checksum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IP address of HA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Care-of address COA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ver.</th>
<th>IHL</th>
<th>TOS</th>
<th>length</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IP identification</th>
<th>flags</th>
<th>fragment offset</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TTL</th>
<th>min. encap.</th>
<th>IP checksum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IP address of HA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>care-of address COA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IP address of MN</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IP address of CN</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TCP/UDP/ ... payload</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

Minimal Encapsulation

- optional
- avoids repetition of identical fields such as TTL, IHL, version, TOS
- only applicable for unfragmented packets, no space left for fragment identification

<table>
<thead>
<tr>
<th>ver.</th>
<th>IHL</th>
<th>TOS</th>
<th>length</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IP identification</th>
<th>flags</th>
<th>fragment offset</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TTL</th>
<th>IP checksum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IP address of HA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>care-of address COA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>lay. 4 prot.</th>
<th>S</th>
<th>reserved</th>
<th>IP checksum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IP address of MN</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IP address of CN</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(only if S=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TCP/UDP/ ... payload</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

Data transfer from the mobile system

1. MN sends to FA
2. FA tunnels packets to HA by encapsulation
3. HA forwards the packet to the receiver (standard case)

Reverse tunneling (RFC 2344)

1. MN sends to FA
2. FA tunnels packets to HA by encapsulation
3. HA forwards the packet to the receiver (standard case)
Mobile IP with reverse tunneling

- Router accept often only “topologically correct” addresses (firewall!)
  - a packet from the MN encapsulated by the FA is now topologically correct
  - furthermore multicast and TTL problems solved (TTL in the home network correct, but MN is too far away from the receiver)
- Reverse tunneling does not solve
  - problems with firewalls, the reverse tunnel can be abused to circumvent security mechanisms (tunnel hijacking)
  - optimization of data paths, i.e. packets will be forwarded through the tunnel via the HA to a sender (double triangular routing)
- Reverse tunneling is backwards compatible
  - the extensions can be implemented easily and cooperate with current implementations without these extensions

Optimization of packet forwarding

- **Triangular Routing**
  - sender sends all packets via HA to MN
  - higher latency and network load
- **“Solutions”**
  - sender learns the current location of MN
  - direct tunneling to this location
  - HA informs a sender about the location of MN
  - big security problems
- **Change of FA**
  - packets on-the-fly during the change can be lost
  - new FA informs old FA to avoid packet loss, old FA now forwards remaining packets to new FA
  - this information also enables the old FA to release resources for the MN

Change of foreign agent

<table>
<thead>
<tr>
<th>CN</th>
<th>HA</th>
<th>FA_{old}</th>
<th>FA_{new}</th>
<th>MN</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>request</td>
<td>update</td>
<td>ACK</td>
</tr>
<tr>
<td></td>
<td></td>
<td>data</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>registration</td>
<td>update</td>
<td>registration</td>
</tr>
<tr>
<td></td>
<td></td>
<td>data</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>update</td>
<td>ACK</td>
<td>data</td>
</tr>
<tr>
<td></td>
<td></td>
<td>data</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>data</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Location services

- **Service that maps node names to (geographic) coordinates**
  - Should be distributed (no require for specialized hardware)
  - Should be efficient
- **Lookup of the position (or COA) of a mobile node**
  - Mobile IP: Ask home agent
  - Home agent is determined through IP (unique ID) of MN
  - Possibly long detours even though sender and receiver are close
  - OK for Internet applications, where latency is (normally) low
- **Other application: Routing in a MANET**
  - MANET: mobile ad hoc network
  - No dedicated routing hardware
  - Limited memory on each node: cannot store huge routing tables
  - Nodes are mostly battery powered and have limited energy
  - Nodes route messages, e.g. using georouting
Home based georouting in a MANET

- **How can the sender learn the current position of another node?**
  - Flooding the entire network is undesirable (traffic and energy overhead)
- **Home based approach**
  - Similar to Mobile IP, each node has a home node, where it stores and regularly updates its current position
  - The home is determined by the unique ID of the node \( t \). One possibility is to hash the ID to a position \( p_t \) and use the node closest to \( p_t \) as home.
  - Thus, given the ID of a node, every node can determine the position of the corresponding home.

### Home based routing

1. Route packet to \( h_t \), the home of the destination \( t \)
2. Read the current position of \( t \)
3. Route to \( t \)

### Home based location service – how good is it?

- Visiting the home of a node might be wasteful if the sender and receiver happen to be close, but the home far away
- The routing **stretch** is defined as
  \[
  \text{stretch} = \frac{\text{length of route}}{\text{length of optimal route}}
  \]
  We want routing algorithms with low stretch.
- Simultaneous message routing and node movement might cause problems
- **Can we do better?**

Classification of location services

- **Proactive**
  - Mobile node divulges its position to all nodes whenever it moves
  - E.g. through flooding
- **Reactive**
  - Sender searches mobile host only when it wants to send a message
  - E.g. through flooding
- **Hybrid**
  - Both, proactive and reactive.
  - Some nodes store information about where a node is located
  - Arbitrarily complicated storage structures
  - Support for simultaneous routing and node mobility

Location services: Lookup & Publish

- **Any node \( A \) can invoke to basic operations:**
  - Lookup\((A, B)\): \( A \) asks for the position of \( B \)
  - Publish\((A, x, y)\): \( A \) announces its move from position \( x \) to \( y \)
- **Open questions**
  - How often does a node publish its current position?
  - Where is the position information stored?
  - How does the lookup operation find the desired information?
The Grid Location Service (GLS), Li et. al (2000)

- Cannot get reasonable stretch with one single home. Therefore, use several homes (location servers) where the node publishes its position.

- The location servers are selected based on a grid structure:
  - The area in which the nodes are located is divided into squares
  - All nodes agree on the lower left corner (0,0) and upper right corner \((2^M, 2^M)\), which forms the square called level-M
  - Recursively, each level-N square is split into 4 level-(N-1) squares
  - The recursion stops for level-1

Addressing of nodes

- **Unique IDs** are generated for each node (e.g. by using a hash-function)
- ID space (all possible hash values) is **circular**
- Every node can find a least greater node w.r.t. the ID space (the closest node)

- Example:
  Let the ID space range from 1 to 99 and consider the IDs \{3, 43, 80, 92\}. Then, the least greater node with respect to the given ID space is
  \[3 \rightarrow 43; 43 \rightarrow 80; 80 \rightarrow 92; 92 \rightarrow 3\]

Selecting location servers

- Each node A recruits location servers using the underlying grid:
  - In each of the 3 level-1 squares that, along with A, make up a level-2 square, A chooses the node closest to its own ID as location server.
  - The same selection process is repeated on higher level squares.

Example for node 92, which selects the nodes \{23, 17, 11\} on the level-1 and \{2, 3, 31\} on level-2.
**Lookup Example**

**Lookup for 17 from 76, 39 and 90**

**Analysis of GLS**

- **Theorem 1**: A query needs no more than $k$ location query steps to reach a location server of the destination when the sender and receiver are colocated in a level-$k$ square.

- **Theorem 2**: The query never leaves the level-$k$ square in which the sender and destination are colocated.

---

**Complete example**

**Querying location of other nodes**

- **Lookup($A, B$): Find a location server of node $B$**
  1. Node $A$ sends the request (with georouting) to the node with ID closest to $B$ for which $A$ has location information.
  2. Each node on the way forwards the request in the same way.
  3. Eventually, the query reaches a location server of $B$, which forwards it to $B$.

Example: Send packet from 81 to 23.
GLS has no worst case guarantees

- The lookup cost between two nodes might be arbitrarily high even though the nodes are very close.
- The publish cost might be arbitrarily high even though a node only moved a very short distance.
- In sparse networks, routing to the location server may have worst case cost; while routing directly can be more efficient.

GLS and mobility

- Node crosses boundary line: what happens to the node’s role as location server?
  - Must redistribute all information in the old level
  - Gather new information in the new level
  - Publish cost is arbitrarily high compared to the moved distance
- A lookup happening in parallel with node movement might fail. Thus, GLS does not guarantee delivery for real concurrent systems, where nodes might move independently at any time.

Improving GLS

- Goals for MLS
  - Publish cost only depends on moved distance
  - Lookup cost only depends on the distance between the sender and receiver
  - Nodes might move arbitrarily at any time, even while other nodes issue lookup requests
  - Determine the maximum allowed node speed under which MLS still guarantees delivery

Location pointers (aka location servers)

- Difference to GLS:
  - Only one location pointer (LP) per level (L) (GLS: 3 location servers)
  - The location pointer only knows in which sub-level the node is located (GLS: the location server knows the exact position)
Location pointer & Notation

- **Notation:**
  - $LP^t_k$: Location pointer for node $t$ on level-$k$
  - $L^t_k$: Level-$k$ that contains node $t$

- The location pointers are placed depending on their ID, as in the home-based lookup system.

- The position of $LP^t_k$ is obtained by hashing the ID of node $t$ to a position in $L^t_k$. The location pointer is stored on the nearest nodes.

Routing in MLS

- Routing from a node $s$ to a node $t$ consists of two phases:
  1. Find a location pointer $LP^t_k$
  2. Once a first location pointer is found on level-$k$, we know in which of the 4 sub-squares $t$ is located and thus in which $L^t_{k-1}$ $t$ has published another location pointer $LP^t_{k-1}$.

Recursive, the message is routed towards location pointers on lower levels until it reaches the lowest level, from where it can be routed directly to $t$.

Routing in MLS (2)

- When a node $s$ wants to find a location pointer of a node $t$, it first searches in its immediate neighborhood and then extends the search area with exponentially growing coverage.
  - First, try to find a location pointer $LP^t_0$ in $L^t_0$ or one of its 8 neighboring levels.
  - Repeat this search on the next higher level until a $LP^t_k$ is found.

- The lookup path draws a spiral-like shape with exponentially increasing radius until it finds a location pointer of $t$.

- Once a location pointer is found, the lookup request knows in which sub-square it can find the next location pointer of $t$.

Support for mobility in MLS

- A location pointer only needs to be updated when the node leaves the corresponding sub-square.
  - $LP^t_2$ is OK as long as $t$ remains in the shaded area.
  - Most of the time, the closest few location pointers need to be updated due to mobility.

- Not enough: If a node moves across a level boundary, many pointers need to be updated. E.g. a node oscillates between the two points $a$ and $b$.
Lazy publishing

- Idea: Don’t update a level pointer $LP^t_k$ as long as $t$ is still somewhat close to the level $L_k$ where $LP^t_k$ points.

- Breaks the lookup: $LP^t_{i+1}$ points to a level that does not contain $LP^t_i$.

Lazy publishing with forwarding pointers

- No problem, add a **forwarding pointer** that indicates in which neighboring level the location pointer can be found.

Concurrency in MLS

- Allowing for concurrent lookup requests and node mobility is somewhat tricky, especially the deletion of pointers.

- Note that a lookup request needs some time to travel between location pointers. The same holds for requests to create or delete location (or forwarding) pointers.

- Example:
  - A lookup request follows $LP^t_{i+1}$ and node $t$ moves as indicated
  - $t$ updates its $LP^t_i$ and $LP^t_{i+1}$ and removes the $FP^t_i$ and the old $LP^t_i$
  - The lookup request fails if it arrives after the $FP^t_i$ has been removed

Concurrency in MLS (2)

- No problem either: Instead of removing a location pointer or forwarding pointer, replace it with a **temporary pointer** that remains there for a short time until we are sure that no lookup request might arrive anymore on this outdated path.

- Similar to the forwarding pointer, a temporary pointer redirects a lookup to the neighbor level where the node is located.
Properties of MLS

- Constant lookup stretch
  - The length of the chosen route is only a constant longer than the optimal route
- Publish cost is $O(d \log d)$ where moved distance is $d$
  - Even if nodes move considerably, the induced message overhead due to publish requests is moderate.
- Works in a concurrent setup
  - Lookup requests and node movement might interleave arbitrarily
- Nodes might not move faster than $1/15$ of the underlying routing speed
  - We can determine the maximum node speed that MLS supports. Only if nodes move faster, there might arise situations where a lookup request fails.

MLS Conclusions

- It's somewhat tricky to handle concurrency properly
  - Use of temporary forwarding pointers
- MLS is the first location service that determines the maximum speed at which nodes might move
  - Without the speed limitation, no delivery guarantees can be made!
- Drawbacks
  - MLS utilizes an underlying routing algorithm that can deliver messages with constant stretch given the position of the destination
  - MLS requires a relatively dense node population

Location service of GSM: elements and interfaces

Architecture of the GSM system

- GSM is a PLMN (Public Land Mobile Network)
- several providers setup mobile networks following the GSM standard within each country
- components
  - MS (mobile station)
  - BS (base station)
  - MSC (mobile switching center)
  - LR (location register)
- subsystems
  - RSS (radio subsystem): covers all radio aspects
  - NSS (network and switching subsystem): call forwarding, handover, switching
  - OSS (operation subsystem): management of the network
GSM: Mobile Terminated Call

1: calling a GSM subscriber
2: forwarding call to GMSC
3: signal call setup to HLR
4, 5: request MSRN from VLR
6: forward responsible MSC to GMSC
7: forward call to current MSC
8, 9: get current status of MS
10, 11: paging of MS
12, 13: MS answers
14, 15: security checks
16, 17: set up connection

GSM: Mobile Originated Call

1, 2: connection request
3, 4: security check
5-8: check resources (free circuit)
9-10: set up call

GSM signaling from/to MS in MTC and MOC

<table>
<thead>
<tr>
<th>MS</th>
<th>MTC</th>
<th>BTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>paging request</td>
<td>channel request</td>
<td>immediate assignment</td>
</tr>
<tr>
<td>immediate assignment</td>
<td>paging response</td>
<td>authentication request</td>
</tr>
<tr>
<td>authentication request</td>
<td>authentication response</td>
<td>ciphering command</td>
</tr>
<tr>
<td>ciphering command</td>
<td>ciphering complete</td>
<td>setup</td>
</tr>
<tr>
<td>setup</td>
<td>call confirmed</td>
<td>assignment command</td>
</tr>
<tr>
<td>assignment command</td>
<td>assignment complete</td>
<td>alerting</td>
</tr>
<tr>
<td>alerting</td>
<td>connect</td>
<td>connect acknowledge</td>
</tr>
<tr>
<td>connect</td>
<td>data/speech exchange</td>
<td>data/speech exchange</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MS</th>
<th>MOC</th>
<th>BTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>channel request</td>
<td>immediate assignment</td>
<td>service request</td>
</tr>
<tr>
<td>immediate assignment</td>
<td>authentication request</td>
<td>authentication response</td>
</tr>
<tr>
<td>authentication response</td>
<td>ciphering command</td>
<td>ciphering complete</td>
</tr>
<tr>
<td>ciphering complete</td>
<td>setup</td>
<td>call confirmed</td>
</tr>
<tr>
<td>setup</td>
<td>assignment command</td>
<td>assignment complete</td>
</tr>
<tr>
<td>assignment complete</td>
<td>alerting</td>
<td>connect</td>
</tr>
<tr>
<td>alerting</td>
<td>connect acknowledge</td>
<td>data/speech exchange</td>
</tr>
<tr>
<td>connect</td>
<td>data/speech exchange</td>
<td>data/speech exchange</td>
</tr>
</tbody>
</table>

Mobility Models

- When studying mobility, one might resolve to models. We have:
- Worst-case model with maximum speed restrictions (as seen)
- Census-based models
  - Street maps, augmented with info what kind of people live/work/shop... in which areas, and how do they move between these areas
  - This is also popular in research about viruses and diseases
- Random models
  - Brownian motion
  - Random waypoint
  - Random trip
The Random Waypoint Model

In its simplest form:
- Mobile picks next waypoint $M_n$ uniformly in area
- Mobile picks next speed $V_n$ uniformly in $[v_{min}; v_{max}]$
- Both independent of past and present
- Mobile moves towards $M_n$ at constant speed $V_n$

The Random Trip model

- Random Waypoint is a special case of Random Trip:
  - Mobile picks a path in a set of paths and a speed
  - At end of path, mobile picks a new path and speed
  - Mobiles may decide to wait and sleep at destinations before going on the next leg
  - E.g. shortest Euclidean path in non-convex area, or shortest path on street map

Simulation problems

- If you simulate mobility, you need to take care about system leveling off...
- The problem is that the steady-state is in the infinite...
Simulation “solutions”

- The problem is that your simulations may show results which differ from "reality".

- A simple rule of thumb (which is wrong, but somehow "acceptable"): If you want to simulate for time T, you really need to simulate time 2T, and throw the first half of your simulation away.

- Another (also wrong) solution is to start each node at a position p which is uniformly random between uniformly random points s and t, and with velocity according to the distribution 1/v. However, also this is not correct...

Simulation solution

- The correct solution is to simultaneously draw position and velocity from the steady-state distribution (see work by Le Boudec for details.)

Open problem

- Even systems like MLS still make way too many simplifying assumptions. So there is the obvious question about a location service which is practical.

- Essentially a good location service system needs to
  1. work in dynamic environments
  2. give acceptable memory and communication loads
  3. provide stretch guarantees
  4. neither make funny assumptions about node distributions...
  5. … nor about mobility patterns
  6. be secure

Figure 9: Density of the time-stationary distribution of (a) node location f (r) and (b) the previous (or next) waypoint. The contour plots (a) and (b) show lines of equal density. The difference in density between two adjacent lines is 0.025. In (a) the density decreases from 0.5523 (center) to 0 (edge); in (b) it increases from 0.1834 to 0.3686. The density in (b) appears to have circular symmetry, but this is not exactly true: (c) plots, for the same case as (b), the density as a function of r for points of the form x = 0.5 cos(θ), y = 0.5 sin(θ) (i.e., points that are at a fixed distance r = 0.5 from the center).