Eidgenössische Technische Hochschule Zürich
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Distributed Computing Group

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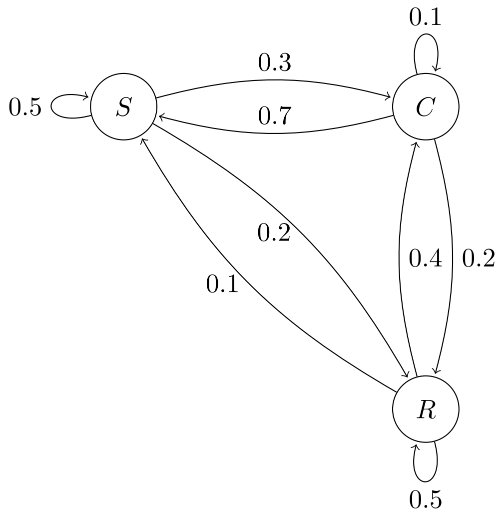
Discrete Event Systems
Solution to Exercise 10

1 The Winter Coat Problem

a) The following Markov chain models the weather situation of Robinson's island.



b) We need to determine the expected hitting time h_{SS}. Using the formula of slide 35, we obtain the following equation system:

$$h_{SS} = 1 + 0.3h_{CS} + 0.2h_{RS} \tag{1}$$
$$h_{CS} = 1 + 0.1h_{CS} + 0.2h_{RS} \tag{2}$$
$$h_{RS} = 1 + 0.4h_{CS} + 0.5h_{RS} \tag{3}$$

(1) and (2) yield that $h_{CS} = \frac{5}{6}h_{SS}$, from (1) and (3) we obtain that $h_{RS} = \frac{40}{23}h_{SS} - \frac{10}{23}$. Setting these results into (1), we obtain

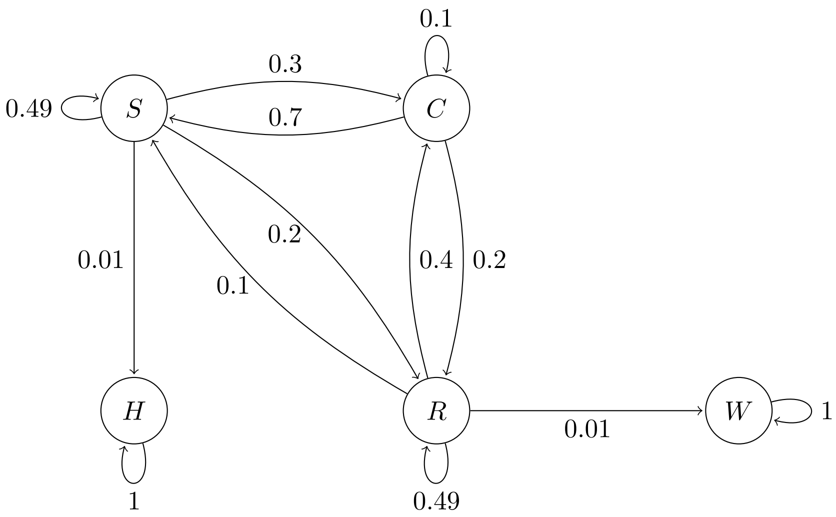
$$h_{SS} = 1 + 0.3\left(\frac{5}{6}h_{SS}\right) + 0.2\left(\frac{40}{23}h_{SS} - \frac{10}{23}\right)$$

Solve for h_{SS} to obtain

$$h_{SS} = \frac{1 - \frac{2}{23}}{1 - \frac{1}{4} - \frac{8}{23}} = \frac{84}{37} \approx 2.27$$

Thus, Mr. Robinson has to wait 2.27 days (in expectation) until having again a sunny day.

c) The modified Markov chain looks as following:



d) We need to determine the arrival probability f_{SW}, the probability that the weather will turn to winter. Using the formula of slide 35, we obtain the following equation system:

$$\begin{aligned}
f_{SW} &= 0 + 0.3f_{CW} + 0.2f_{RW} + 0.49f_{SW} + 0.01f_{HW} && (4)\\
f_{CW} &= 0 + 0.7f_{SW} + 0.2f_{RW} + 0.1f_{CW} && (5)\\
f_{RW} &= 0.01 + 0.4f_{CW} + 0.1f_{SW} + 0.49f_{RW} && (6)\\
f_{HW} &= 0 && (7)
\end{aligned}$$

Solving the equation system yields

$$f_{SW} = \frac{240}{619},\ f_{RW} = \frac{249}{619},\ f_{CW} = \frac{242}{619}$$

And therefore, the probability that the weather turns to winter (snowing) and Mr. Robinson needs a winter coat is $\frac{240}{619} \approx 0.39$. Note that $f_{SH} = 1 - f_{SW} = \frac{379}{619}$.

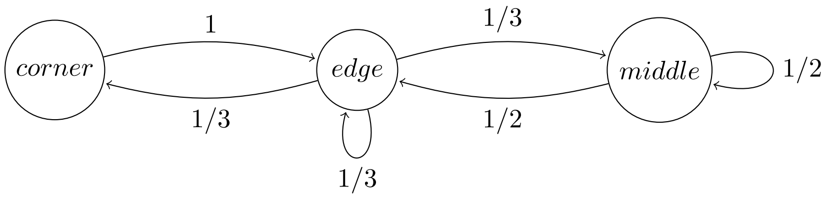
2 Probability of Arrival

The proof is similar to the one about the transition time h_{ij} (see script). We express f_{ij} as a condition probability that depends on the result of the first step in the Markov chain. Recall that the random variable T_{ij} is the *hitting time*, that is, the number of steps from i to j. We get $Pr[T_{ij} < \infty | X_1 = k] = Pr[T_{ij} < \infty]$ for $k \neq j$ and $Pr[T_{ij} < \infty | X_1 = j] = 1$. We can therefore write f_{ij} as

$$\begin{aligned}
f_{ij} &= Pr[T_{ij} < \infty] = \sum_{k \in S} Pr[T_{ij} < \infty | X_1 = k] \cdot p_{ik}\\
&= p_{ij} \cdot Pr[T_{ij} < \infty | X_1 = j] + \sum_{k \neq j} Pr[T_{ij} < \infty | X_1 = k] \cdot p_{ik}\\
&= p_{ij} + \sum_{k \neq j} p_{ik} f_{kj}.
\end{aligned}$$

3 Night Watch

a) Observe that the problem is symmetric, e.g., from all four corners, the situation looks the same, and the probability of being in a specific corner room is the same for all corners. The same holds for rooms at the border and for rooms in the middle. Thus, instead of using 16 states, we consider the following simplified Markov chain consisting of 3 states only:



Hence, in the steady state, it holds that

$$P_c = 1/3 \cdot P_e; \quad P_e = 1/3 \cdot P_e + 1/2 \cdot P_m + P_c; \quad 1 = P_c + P_e + P_m$$

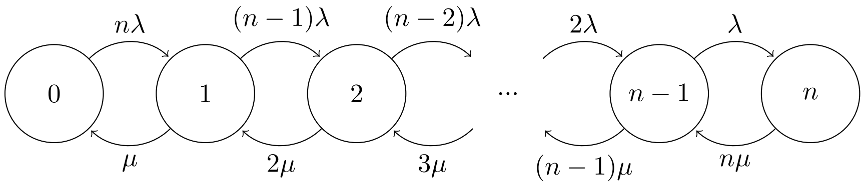
Solving this equation system gives: $P_c = 1/6$. The probability of being in a specific corner is therefore $1/6 \cdot 1/4 = 1/24$.

b) Since the two walks are independent, we have

$$1/24 + 1/24 - (1/24)^2 = 0.082.$$

4 "Hopp FCB!"

a) We know that the minimum of i independent and exponentially distributed (with parameter λ) random variables is an exponentially distributed random variable with parameter $i\lambda$. Thus, we have the following birth-death-process:



b) Let p_i be the probability of state i in the equilibrium. In a general birth-death-process with transition parameters λ_i and μ_i, it holds that

$$p_1\mu_1 = p_0\lambda_1 \Rightarrow p_1 = \frac{\lambda_1}{\mu_1}p_0.$$

By induction, we have

$$p_{i+1} \cdot \mu_{i+1} + p_{i-1} \cdot \lambda_i = p_i \cdot (\lambda_{i+1} + \mu_i)$$

and thus

$$p_i = \frac{\lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_i}{\mu_1 \cdot \mu_2 \cdot \cdots \cdot \mu_i} p_0.$$

Applying this formula to our process yields

$$p_i = \frac{n(n-1) \cdot \cdots \cdot (n-i+1) \cdot \lambda^i}{1 \cdot 2 \cdot \cdots \cdot i \cdot \mu^i} p_0 = \binom{n}{i} \left(\frac{\lambda}{\mu}\right)^i p_0.$$

Let $\rho := \frac{\lambda}{\mu}$. Since the sum of all probabilities equals 1, we have

$$p_0 \sum_{i=0}^{n} \binom{n}{i} \rho^i = p_0(1+\rho)^n = 1 \Rightarrow p_0 = \frac{1}{(1+\rho)^n}.$$

Finally,

$$p_i = \frac{\binom{n}{i}\rho^i}{(1+\rho)^n}.$$

c) A team is able to play if and only if there are at least eleven fit players:

$$p_{11} + p_{12} + \cdots + p_{20} = 0.965.$$

Thus, the FCB team has enough players that it can participate in most of the matches (probability > 95 %).