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Discrete Event Systems

Solution to Exercise Sheet 10

1 Labelled Graphs

a) We give two algorithms, an iterative and a recursive one, that calculate whether the given LTS $\mathcal{L}$ accepts the word $\omega = w_1 \dots w_n$.

Algorithm 1 ACCEPTITERATIVE($\mathcal{L}, \omega$)

```
Input: LTS L = (S, S_0, Act, E), ω = w_1 w_2 ... w_n
states ← S_0                                  ▷ contains the states reachable by w_1 ... w_{i-1}
for i ← 1 to n do
    newStates ← ∅                             ▷ contains the new states reachable by w_1 ... w_i
    for all v ∈ states do                     ▷ For all current states...
        for all c ∈ PostSetNodes(v) do        ▷ For all reachable states...
            if Act((v, c)) = w_i then         ▷ If the label matches...
                newStates ← newStates ∪ {e.target()}   ▷ ...remember the state
    if newStates = ∅ then                     ▷ If no edge with label w_i exists...
        return false
    states ← newStates
return true
```

Algorithm 2 ACCEPTRECURSIVE($\mathcal{L}$, states, ω)

```
Input: LTS L = (S, S_0, Act, E), states: set of states, ω = w_1 w_2 ... w_n
newStates ← ∅
if ω = ∅ then                                 ▷ Every letter of the word has been matched to a path.
    return true
else if states = ∅ then                       ▷ No state was reachable by the last letter.
    return false
for all v ∈ states do                         ▷ For all current states...
    for all c ∈ PostSetNodes(v) do            ▷ For all reachable states...
        if Act((v, c)) = w_1 then             ▷ If the label matches...
            newStates ← newStates ∪ {c}       ▷ ...remember the state
ω ← w_2, ..., w_n                             ▷ Remove first letter of ω
return ACCEPTRECURSIVE(L, newStates, ω)       ▷ Recursive call for the remaining word
```

The initial call is ACCEPTRECURSIVE($\mathcal{L}, \mathbb{S}_0, \omega$).

2 Structural Properties of Petri Nets and Token Game

a) The pre and post sets of a transition are defined as follows:

- pre set: $\bullet t := \{p \mid (p,t) \in C\}$
- post set: $t\bullet := \{p \mid (t,p) \in C\}$,

the pre and post sets of a place are defined analogously.

For the petri net N_1 we obtain the following sets:

$$\begin{aligned} \bullet t_5 &= \{p_5, p_9\}, & t_5\bullet &= \{p_6\} \\ \bullet t_8 &= \{p_8\}, & t_8\bullet &= \{p_{10}, p_5\} \\ \bullet p_3 &= \{t_2\}, & p_3\bullet &= \{t_3\} \end{aligned}$$

b) A transition is enabled if all places in its pre set contain enough tokens. In the case of N_1, which has only unweighted edges, one token per place suffices. When t_2 fires, it consumes one token out of each place in the pre set of t_2 and produces one token on each place in the post set of t_2. Hence, the firing of t_2 produces one token on place p_3 and p_9 each, the one on p_2 is consumed. After this, t_5 is enabled because both p_9 and p_5 hold one token. However, t_3 is not enabled because p_3 contains a token but p_{10} does not.

c) Before t_2 fires there are two tokens in N_1, one on p_2 and p_5 each. Directly afterwards, there are tokens on places p_3, p_9 und p_5.

d) A token traverses the upper cycle until t_2 fires. Then one token remains on p_3 and waits, and another one is produced in p_9, which enables transition t_5. When t_5 consumes the tokens on p_9 and p_5 and produces a token on p_6, this one can traverse the lower cycle until t_8 is enabled. One token now remains on p_5 and waits, another one enables t_3, because there is still one token on p_3. Now one token traverses the upper cycle again until t_2 is enabled, and so on.

Hence, this petri net models two processes which always appear alternately.

The reachability graph $RG(P, \vec{s}_0)$ of a petri net P is a quadruple $(\mathbb{S}, \mathbb{S}_0, Act, \mathbb{E})$ such that

- $\mathbb{S}$ is the set of reachable states of P starting from $\vec{s}_0$
- $\mathbb{S}_0 := \{\vec{s}_0\}$ is the start state of P
- Act is the set of transition labels
- $\mathbb{E} \subseteq \mathbb{S} \times Act \times \mathbb{S}$ is the set of edges such that $\mathbb{E} = \{\big(\vec{s}, t, \delta(\vec{s}, t)\big) \mid \vec{s} \in \mathbb{S} \wedge t \in T \wedge \vec{s} \triangleright t\}$

Usually the states of the petri net are denoted by vectors such that the i-th position in the vector indicates the number of tokens on place p_i of the petri net. So, for example, the starting state $\vec{s}_0$ of N_1, in which the places p_1 and p_5 hold one token each, is denoted by

$\vec{s}_0 = (1,0,0,0,1,0,0,0,0,0)$. Hence, the reachability graph looks as follows:

$$
\begin{aligned}
\mathbb{S} = \{ \quad & (1,0,0,0,1,0,0,0,0,0),(0,1,0,0,1,0,0,0,0,0),(0,0,1,0,1,0,0,0,1,0),\\
& (0,0,1,0,0,1,0,0,0,0),(0,0,1,0,0,0,1,0,0,0),(0,0,1,0,0,0,0,1,0,0),\\
& (0,0,1,0,1,0,0,0,0,1),(0,0,0,1,1,0,0,0,0,0) \quad \},\\[1ex]
\mathbb{S}_0 = \{ \quad & (1,0,0,0,1,0,0,0,0,0) \quad \},\\[1ex]
\mathit{Act} = \{ \quad & t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8,t_9,t_{10} \quad \},\\[1ex]
\mathbb{E} = \{ \quad & \big((1,0,0,0,1,0,0,0,0,0),t_1,(0,1,0,0,1,0,0,0,0,0)\big),\\
& \big((0,1,0,0,1,0,0,0,0,0),t_2,(0,0,1,0,1,0,0,0,1,0)\big),\\
& \big((0,0,1,0,1,0,0,0,1,0),t_5,(0,0,1,0,0,1,0,0,0,0)\big),\\
& \big((0,0,1,0,0,1,0,0,0,0),t_6,(0,0,1,0,0,0,1,0,0,0)\big),\\
& \big((0,0,1,0,0,0,1,0,0,0),t_7,(0,0,1,0,0,0,0,1,0,0)\big),\\
& \big((0,0,1,0,0,0,0,1,0,0),t_8,(0,0,1,0,1,0,0,0,0,1)\big),\\
& \big((0,0,1,0,1,0,0,0,0,1),t_3,(0,0,0,1,1,0,0,0,0,0)\big),\\
& \big((0,0,0,1,1,0,0,0,0,0),t_4,(1,0,0,0,1,0,0,0,0,0)\big) \quad \}.
\end{aligned}
$$

For better legibility we denote the states in such a way that the index contains the places that hold a token in this state, for example $\vec{s}_0 = (1,0,0,0,1,0,0,0,0,0) = s_{1,5}$.

Then the reachability graph can also be specified as follows:

$$
\begin{tikzcd}
\arrow[r] & s_{1,5} \arrow[r, "t_1"] & s_{2,5} \arrow[r, "t_2"] & s_{3,5,9} \arrow[r, "t_5"] & s_{3,6} \arrow[r, "t_6"] & s_{3,7} \arrow[r, "t_7"] & s_{3,8} \arrow[r, "t_8"] & s_{3,5,10} \arrow[r, "t_3"] & s_{4,5} \arrow[lllllll, bend left, "t_4"]
\end{tikzcd}
$$

3 Basic Properties of Petri Nets

A petri net is k-bounded, if there is no fire sequence that makes the number of tokens in one place grow larger than k. It is obvious that petri net N_2 is 1-bounded if $k \leq 1$. This holds because in the initial state there is only one token in the net, and in the case $k \leq 1$ no transition increases the number of tokens in N_2. If $k \geq 2$, the number of tokens in p_1 can grow infinitely large by repeatedly firing t_1, t_3 and t_4. So, the petri net N_2 is unbounded for $k \geq 2$.

A petri net is deadlock free if no fire sequence leads to a state in which no transition is enabled. If $k = 0$, N_2 is not deadlock-free. The fire sequence t_1, t_3, t_4 causes the only existing token to be consumed and hence, there is no enabled transition any more. For $k \geq 1$, however, no deadlock can occur.

4 Reachability Analysis for Petri Nets

a) Petri nets may possess infinite reachability graphs, e.g. N_2 with $k \geq 2$. If the state in question is actually reachable in such a petri net, the reachability algorithm will eventually terminate. If it is not reachable, the algorithm will never be able to determine this with absolute certainty (cf. halting problem).

b) We determine the incidence matrix of the petri net as explained in the lecture.

$$
\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}
$$

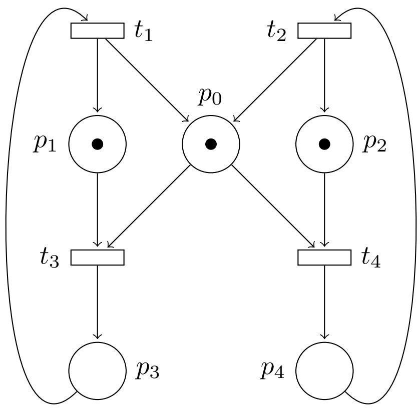
We are interested in whether the state $\vec{s} = (101, 99, 4)$ is reachable from the initial state $\vec{s_0} = (1, 0, 0)$. If the equation system $\mathbf{A} \cdot \vec{f} = \vec{s} - \vec{s_0}$ has no solution, we know that the state $\vec{s}$ is not reachable from s_0. "Unfortunately",

$$\begin{pmatrix} -1 & 1 & 0 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 99 \\ 4 \end{pmatrix}$$

is satisfiable. To show that $\vec{s}$ is reachable from $\vec{s_0}$, we have to give a firing sequence through which we get from $\vec{s_0}$ to $\vec{s}$. From the last equation of the above equation system, we know that $f_3 = f_4 + 4$. Hence, in the desired firing sequence, f_3 is fired four times more than f_4. However, $\vec{f}$ does not tell us about the firing order. Considering the petri net, we can see that – starting from $\vec{s_0}$ – the number of tokens in p_1 increases by one after firing t_1, t_3, and t_4 in this order. Repeating this for 203 times yields the state $(204, 0, 0)$. Firing t_1 for 103 times followed by firing t_3 for four times finally yields state $\vec{s}$.

5 Mutual Exclusion

For each process we introduce two places (p_1, p_2, p_3 und p_4) representing the process within the normal program execution (p_1, p_2) as well as in the critical section (p_3, p_4). For each process, we have a token indicating which section of the program currently is executed. Additionally, we introduce a place p_0 representing the mutex variable. If the mutex variable is 0, then we have a token at p_0. We have to make sure that a process can only enter its critical section if there is a token at the mutex place. The resulting petri net looks as follows.



Assume that initially, both processes are in an uncritical section (in the petri net, this is denoted by a token in place p_1 and p_2 respectively). A process can only enter its critical section (p_3/p_4) if there is a token at p_0. In this case, the token is consumed when entering the critical section. A new mutex token at p_0 is not created until the process leaves its critical section. Hence, both processes exclude each other mutually from the concurrent access to the critical section.