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Discrete Event Systems

Solution to Exercise Sheet 10

1 Colour Blindness

a) The number of colour blind people X in a sample of 100 is binomially distributed where each person is colour blind with probability $p = 0.02$. Hence we have

$$\Pr[X = k] = \binom{100}{k} p^k (1-p)^{100-k} \quad .$$

The probability that at most one person out of 100 is colour blind is given by

$$\begin{aligned}\Pr[X \leq 1] &= \Pr[X = 0] + \Pr[X = 1] \\ &= \binom{100}{0} p^0 (1-p)^{100} + \binom{100}{1} p^1 (1-p)^{100-1} \\ &\approx 0.403\end{aligned}$$

If we assume X to be Poisson-distributed (see **b)**), we get

$$\Pr[X \leq 1] \approx 0.406 \quad .$$

b) Since the sample size n is large and the probability for someone being colour blind is small, we can estimate the distribution of colour blind people with the Poisson distribution.

The Poisson distribution

The Poisson distribution is a *discrete* probability distribution which is applied often to approximate the binomial distribution for large number n of repetitions and small success probability p of the underlying Bernoulli experiments. According to two frequently used rules of thumb, this approximation is good if $n \geq 20$ and $p \leq 0.05$, or if $n \geq 100$ and $np \leq 10$.
The Poisson distribution is often used to estimate the probability of a given number of events occurring in a fixed interval of time and/or space if these events occur with a known average rate and independently of the time since the last event. The parameter $\lambda = np$ of the distribution is the expected number of occurrences in the interval.

$$\Pr[X = x] = \frac{\lambda^x}{x!} e^{-\lambda}$$

Since we expect the sample size n to be larger than 20 and we have $p = 0.02$, we can assume the number X of colour blind persons in a sample of n persons to be Poisson-distributed with parameter $\lambda = np = n/50$. The probability that at least one person is colour blind in a sample of size n is now given by

$$\begin{aligned}
\Pr[X \geq 1] &= 1 - \Pr[X = 0] \\
&= 1 - e^{-\lambda} \cdot \frac{\lambda^0}{0!} \\
&= 1 - e^{-n/50} \quad .
\end{aligned}$$

Solving the inequality $\Pr[X \geq 1] \geq 90\%$ for n yields $n \geq 116$. Hence, in a sample of 116 persons we have at least one colour blind person with probability 90%.

2 Gloriabar

a) You might be tempted to model this situation by a queue with a bounded number of states, because the maximal number of persons in the line is bounded by 540. However, the situation can also be modeled by an infinite M/M/1 queue without losing too much accuracy; the parameter ρ will not be too large, such that the probability to reach the state in which 540 persons are standing in the queue at once is extremely small anyway. The modeling by an infinite M/M/1 queue conveniently allows us to apply Little's Law (slides 76 ff.) and therefore, we can use the formulae for the response and waiting time from slide 79:

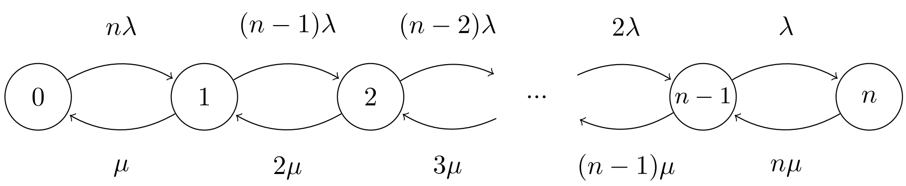
We have an arrival rate of $\lambda = 540/(90 \cdot 60) = 1/10$ (persons per second), and $\mu = 1/9$ (persons per second). Thus $\rho = \lambda/\mu = 9/10$. Applying Little's Law and the mentioned formulae yields: The expected waiting time is $W = \rho/(\mu - \lambda) = 81$ seconds; the expected time until the student has paid for her menu is given by $T = 1/(\mu - \lambda) = 90$ seconds.

b) We use the formula for the expected number of jobs in the queue from slide 79 and obtain queue length of $N_Q = \rho^2/(1-\rho) = 8.1$.

c) We require that $T = 1/(\mu - 0.1) = 90/2$. Thus, $\mu = 11/90$, i.e., instead of 9 secs, the service time should be $90/11 \approx 8.2$ secs.

3 Beachvolleyball

a) We know that the minimum of i independent and exponentially distributed (with parameter λ) random variables is an exponentially distributed random variable with parameter $i\lambda$. Thus, we have the following birth-death-process:



b) Let π_i be the probability of state i in the equilibrium. From slide 87, we know that

$$\pi_i = \pi_0 \cdot \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}}$$

and thus

$$\pi_i = \pi_0 \cdot \frac{\lambda_0 \cdot \lambda_1 \cdot \dots \cdot \lambda_{i-1}}{\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_i}.$$

Applying this formula to our process yields

$$\pi_i = \pi_0 \cdot \frac{n(n-1)\cdot \dots \cdot (n-i+1)\cdot \lambda^i}{1\cdot 2\cdot \dots \cdot i \cdot \mu^i} = \pi_0 \cdot \binom{n}{i} \cdot \rho^i \tag{1}$$

where $\rho := \frac{\lambda}{\mu}$. We know that the sum of all probabilities equals 1, so we have

$$\sum_{i=0}^{n} \pi_i = \pi_0 \sum_{i=0}^{n} \binom{n}{i} \rho^i = 1$$

Using the given formula for the binomial series

$$\sum_{i=0}^{n} \binom{n}{i} x^i = (1+x)^n$$

we obtain

$$\pi_0 (1+\rho)^n = 1 \ .$$

Finally, we obtain

$$\pi_i = \frac{\binom{n}{i}\rho^i}{(1+\rho)^n} \ .$$

c) (i) It is $\rho = 3/9 = 1/3$. We calculate the probability that there are less than two fit players:

$$\begin{aligned}
\pi_0 + \pi_1 &= \frac{1}{(1+\rho)^n} \cdot \left(1 + \binom{n}{1} \cdot \rho^1\right) \\
&= \left(\frac{3}{4}\right)^5 \cdot \left(1 + \frac{5}{3}\right) \\
&= \frac{3^5}{2^{10}} \cdot \frac{8}{3} \\
&= \frac{3^4}{2^7} \approx 0.63
\end{aligned}$$

Thus, the Disco team cannot participate in the tournament with probability 0.63.

(ii) Now, $\rho = 2/4 = 0.5$. Again, we calculate $\pi_0 + \pi_1$:

$$\begin{aligned}
\pi_0 + \pi_1 &= \frac{1}{(1+\rho)^n} \cdot \left(1 + \binom{n}{1} \cdot \rho^1\right) \\
&= \frac{1}{1.5^5} \cdot (1 + 0.5 \cdot 5) \\
&= \frac{2^5 \cdot 3.5}{3^5} \approx 0.46
\end{aligned}$$

Hence, the probability that the Disco team cannot participate is 0.46!

(iii) In general, if $\rho \geq 1$, an M/M/1 queue might grow infinitely and therefore doesn't have a stationary distribution. This cannot happen in this birth-and-death process, though, because there is only a bounded number of states. Hence, the process has a stationary distribution even for $\rho \geq 1$.

4 Theory of Ice Cream Vending

The situation can be described by a classic M/M/2 system. According to slide 90, there is an equilibrium iff

$$\rho = \lambda/(2\mu) < 1 \ .$$

For the stationary distribution, it holds that

$$\begin{aligned}
\pi_0 &= \frac{1}{\left(\sum_{k=0}^{m-1} \frac{(\rho m)^k}{k!}\right) + \frac{(\rho m)^m}{m!(1-\rho)}} \\
&= \frac{1}{\frac{(2\rho)^0}{0!} + \frac{(2\rho)^1}{1!} + \frac{(2\rho)^2}{2!(1-\rho)}} \\
&= \frac{1}{1 + 2\rho + \frac{4\rho^2}{(2(1-\rho))}} \\
&= \frac{1}{1 + 2\rho + \frac{4\rho^2}{2(1-\rho)}} \\
&= \frac{1}{\frac{2(1-\rho)+4\rho(1-\rho)+4\rho^2}{2(1-\rho)}} \\
&= \frac{2(1-\rho)}{2 - 2\rho + 4\rho - 4\rho^2 + 4\rho^2} \\
&= \frac{2(1-\rho)}{2+2\rho} \\
&= \frac{1-\rho}{1+\rho} \quad .
\end{aligned}$$