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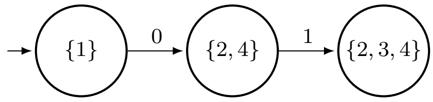
Discrete Event Systems

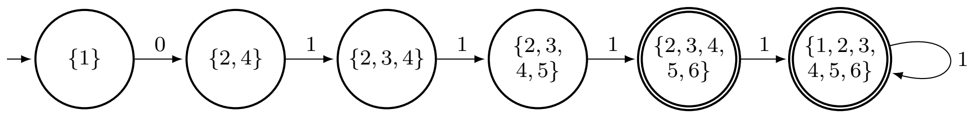
Solution to Exercise Sheet 3

1 Finite Automata and Regular Languages [Exam]

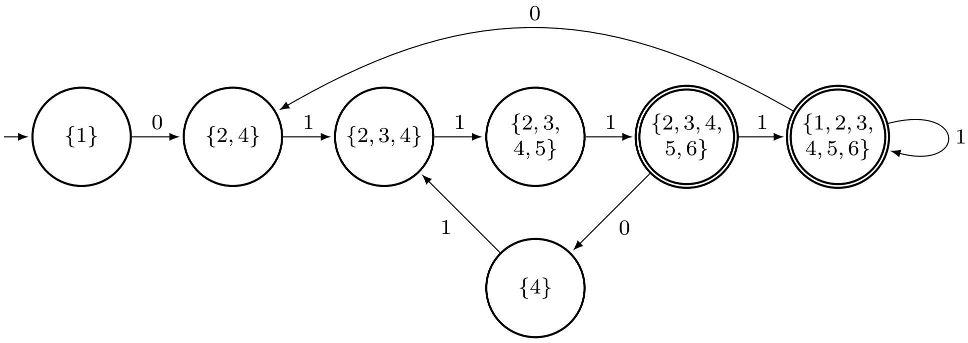
a) We could use the systematic transformation scheme presented in the lecture (slide 1/75). Considering the large number of states, however, this will easily lead to an explosion of states in the derandomized automaton. Hence, we build the deterministic finite automaton in a step-wise manner, only creating those states that are actually required: Initially, the automaton requires a 0. Subsequently, only a 1 is accepted. Including the various transitions, this 1 can lead to three different states, namely states 2, 3, and 4.



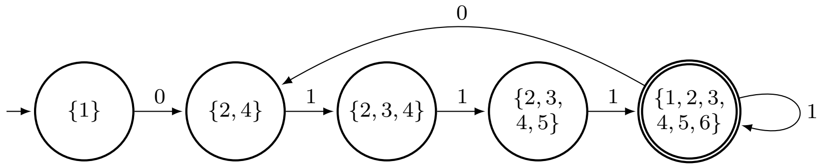
In any of the states 2, 3, and 4, only a 1 is accepted. Assume that the automaton is currently in state 2, this 1 can lead to states $\{2,3,4\}$ when including all ε-transitions. When in state 3, the 1 leads to states $\{2,3,4,5\}$ and finally, when being in state 4, the reachable states given a 1 are $\{2,3,4\}$. Hence, a 1 leads from state $\{2,3,4\}$ to state $\{2,3,4,5\}$. Repeating the same process for state $\{2,3,4,5\}$, we can see that, again, only a 1 is accepted, which leads to state $\{2,3,4,5,6\}$. Because the state 6 in the original NFA was an accepting state, $\{2,3,4,5,6\}$ is also accepting in the DFA. From state $\{2,3,4,5,6\}$, an additional 1 will lead to another accepting state $\{1,2,3,4,5,6\}$. And from this state, any subsequent 1 returns to state $\{1,2,3,4,5,6\}$ as well.



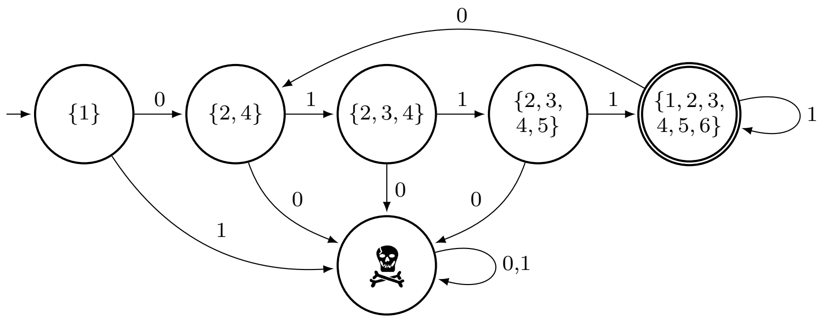
What happens if a 0 occurs in the input? This is feasible only when the deterministic state includes either state 1 or state 6. In state $\{2,3,4,5,6\}$, a 0 necessarily leads to state $\{4\}$, whereas in state $\{1,2,3,4,5,6\}$ a 0 leads to state $\{2,4\}$. In both of these states, the only acceptable input symbol is a 1 and leads to the state $\{2,3,4\}$. Hence, the deterministic finite automaton looks like this:



It can easily be seen, that first the states $\{4\}$, $\{2, 4\}$ and then the states $\{2, 3, 4, 5, 6\}$, $\{1, 2, 3, 4, 5, 6\}$ can be merged and hence, the automaton can be reduced to the one shown in the next figure.



This is not a DFA yet, because the crash state is still missing. The final deterministic automaton looks like this:



b) By studying the above automaton, it can be seen that the following regular language is accepted: $01111^*(01111^*)^* = (0111^+)^+$.

2 Pumping Lemma [Exam]

> **The Pumping Lemma in a Nutshell**
>
> Given a language L, assume for contradiction that L is regular and has the pumping length p. Construct a suitable word $w \in L$ with $|w| \geq p$ ("there *exists* $w \in L$") and show that for *all* divisions of w into three parts, $w = xyz$, with $|x| \geq 0$, $|y| \geq 1$, and $|xy| \leq p$, there *exists* a pumping exponent $i \geq 0$ such that $w' = xy^iz \notin L$. If this is the case, L is not regular.

a) Language L_1 can be shown to be non-regular using the pumping lemma. Assume for contradiction that L_1 is regular and let p be the corresponding pumping length. Choose w to be the word 0110^p1^p. Because w is an element of L_1 and has length more than p, the pumping lemma guarantees that w can be split into three parts, $w = xyz$, where $|xy| \leq p$ and for any $i \geq 0$, we have $xy^iz \in L_1$. In order to obtain the contradiction, we must prove that for every possible partition into three parts $w = xyz$ where $|xy| \leq p$, the word w cannot be pumped. We therefore consider the various cases.

(1) If y starts with any suffix of the first three symbols (i.e. 011) of w, the word w cannot be pumped without violating either the constraints $a = 1$ or $b = 2$ (e.g. $01\,01\,1\,0^p\,1^p$ for $y = 01$) or creating a word with an illegal structure (e.g. $011\,011\,0^p\,1^p$ for $y = 011$).

(2) If y consists of only 0s from the second block, the word $w' = xyyz$ has more 0s than 1s in the last $|w'| - 3$ symbols and hence $c \neq d$.

Note that y cannot contain 1s from the second block because of the requirement $|xy| \leq p$.

We have shown that for all possible divisions of w into three parts, the pumped word is not in L_1. Therefore, L_1 cannot be regular and we have a contradiction.

b) With the adapted language L_2, the proof of non-regularity is much more tricky! Specifically, non-regularity of L_2 cannot be proven using the pumping lemma, because any word in L_2 can actually be pumped! Consider for instance a word w of the form 0110^p1^p. In this case, we can split w into the three parts $x = 0, y = 11, z = 0^p1^p$, which is in accordance with the rules of the pumping lemma. It can be seen, however, that any word xy^iz is also in L_2! That is, the language L_2 can be pumped and yet, it is not regular as shown below.

Assume for contradiction that there exists a finite automaton A which accepts the language L_2. Every word that starts with the input-sequence 0110 is only accepted if the remainder of the word has the form $0^{c-1}1^c$ for some integer $c > 0$. Let q_1 be the state reached after the input 0110. Given the automaton A, we can construct a regular automaton A' that is equivalent to A with the only difference that its initial state is q_1. By the definition of A, this adapted finite automaton A' accepts all words of the form $0^{c-1}d^c$. However, as shown on slide 1/95 of the script, the language $0^{c-1}d^c$ is not regular. Hence, A' and thus A cannot be finite automata. Because there exists a finite automaton for every regular language, it follows that L_2 cannot be regular. Language L_2 shows that while every regular language can be pumped according to the pumping lemma, there are also non-regular languages that can be pumped.

Variant: We can alternatively use the fact that if two languages L and L' are regular, the language defined by the intersection of the two languages $L \cap L'$ is regular as well (cf. p. 1/41). Consider the regular language $L_3 = \{w \in 0110^*1^*\}$. Notice that the intersection of L_3 with $L_2 = \{0^a1^b0^c1^d \mid a, b, c, d \geq 0 \text{ and if } a = 1 \text{ and } b = 2 \text{ then } c = d\}$ contains exactly all words $w \in \{0110^n1^n \mid n \geq 0\}$. This, however, is the exact language L_1 we proved not to be regular in the first part of this exercise. If we assume L_2 to be regular, L_1 must be regular as well, since $L_1 = L_2 \cap L_3$. This is a contradiction. Thus L_2 cannot be regular.

Be Careful!

The argumentation above is based on the closure properties of regular languages and only works in the direction presented. That is, for an operator $\diamond \in \{\cup, \cap, \bullet\}$, we have:

If L_1 and L_2 are regular, then $L = L_1 \diamond L_2$ is also regular.

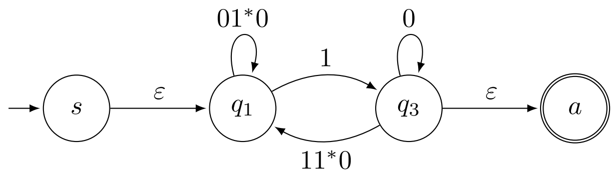
If either L_1 or L_2 or both are non-regular, we cannot deduce the non-regularity of L or vice-versa. Moreover, L being regular does not imply that L_1 and L_2 are regular as well. This may sound counter-intuitive which is why we give examples for the three operators.

- $L = L_1 \cup L_2$: Let L_1 be any non-regular language and L_2 its complement. Then $L = \Sigma^*$ is regular.
- $L = L_1 \cap L_2$: Let L_1 be any non-regular language and L_2 its complement. Then $L = \emptyset$ is regular.
- $L = L_1 \bullet L_2$: Let $L_1 = \{a^*\}$ (a regular language) and $L_2 = \{a^p \mid p \text{ is prime}\}$ (a non-regular language) then $L = \{aaa^*\}$ is regular.

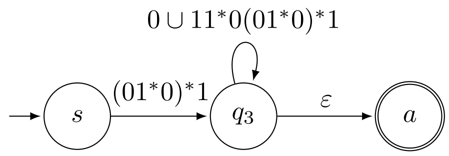
Hence, to prove that a language L_x is non-regular, you assume it to be regular for contradiction. Then you combine it with a *regular* language L_r to obtain a language $L = L_x \diamond L_r$. If L is non-regular, L_x could not have been regular either.

3 Transforming Automata [Exam]

a) The regular expression can be obtained from the finite automaton using the transformation presented in the script on slide 1/85. After ripping out state q_2, the corresponding GNFA looks like this:



After also removing state q_1, the GNFA looks as follows.



Eliminating the last state q_3 yields the final solution, which is $(01^*0)^*1(0 \cup 11^*0(01^*0)^*1)^*$.

Note: Ripping out the interior states in a different order yields a distinct yet equivalent regular expression. The order q_3, q_2, q_1, for example, results in $((0 \cup 10^*1)1^*0)^*10^*$.

b) The best way to solve this problem is to ask, which words are actually not in $\Phi(L)$. The word 1, for instance must be in $\Phi(L)$, because the word 10 is in L. Moreover, the word 11 is in $\Phi(L)$, because 1101 is in L. Also, 10, 01, and 00 are in $\Phi(L)$ because of the words 1000, 0101, and 0010, respectively. More generally, it can be seen from every state in the automaton and for all $k \geq 2$, there is a sequence of k symbols that lead to the accepting state. Hence, all words of length at least 2 are in $\Phi(L)$. Also, as seen before, the word 1 is in $\Phi(L)$. The only words that are not in $\Phi(L)$ are therefore 0 and ε: 0 is not in $\Phi(L)$, because there is no word of length 2 in L starting with 0 that leads to an accepting state, and ε is not in $\Phi(L)$, because $\varepsilon \notin L$. With this, constructing the resulting DFA is now easy.