Discrete Event Systems
Solution to Exercise Sheet 6

1 Soccer Betting

a) The following Markov chain models the different transition probabilities (W:Win, T:Tie, L:Loss):

```
W 0.6 0.3 0.2
T 0.4 0.4 0.2
L 0.7 0.3 0.7
```

b) The transition matrix \( P \) is

\[
P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}.
\]

As you might have noticed, we gave redundant information here. You only need the information that the FCB lost its last game. Thus, the Markov chain is currently in the state \( L \) and hence, the initial vector is \( q_0 = (0 \ 0 \ 1) \). The probability distribution \( q_2 \) for the game against the FC Zurich is therefore given by

\[
q_2 = q_0 \cdot P^2 = (q_0 \cdot P) \cdot P = (0.1 \ 0.2 \ 0.7) \cdot \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{pmatrix} = (0.19 \ 0.24 \ 0.57)
\]

(Note that \( q_0 \) must be a row vector, not a column vector.)

**Hint:** We exploited the associativity of the matrix multiplication to avoid having to calculate \( P^2 \) explicitly. This is usually a good “trick” to avoid extensive and error-prone calculations if no calculator is at hand (as for example in an exam situation \( \bowtie \)).

Given the quotas of the exercise, the expected return for each of the three possibilities (\( W, T, L \)) calculates as follows.

\[
E[W] = 0.19 \cdot 3.5 = 0.665 \\
E[T] = 0.24 \cdot 4 \quad = 0.96 \\
E[L] = 0.57 \cdot 1.5 = 0.855
\]
Therefore, the best choice is not to bet at all since the expected return is smaller than 1 for every choice. If a “sales representative” of the Swiss gambling mafia were to force you to bet, you would be best off with betting on a tie, though.

c) The new Markov chain model looks like this. In addition to the three states $W$, $T$, and $L$, there is now a new state $LL$ which is reached if the team has lost twice in a row.

![Diagram of Markov Chain](image_url)

The new transition matrix $P$ is

$$ P = \begin{pmatrix}
0.6 & 0.2 & 0.2 & 0 \\
0.3 & 0.4 & 0.3 & 0 \\
0.1 & 0.2 & 0 & 0.7 \\
0.05 & 0.1 & 0 & 0.85
\end{pmatrix}. $$

As the FCB has and lost its last two games, the Markov chain is currently in the state $q_0 = (0 \ 0 \ 0 \ 1)$. The probabilities for the game against the FC Zurich can again be computed as follows.

$$ q_3 = q_0 \cdot P^2 = (q_0 \cdot P) \cdot P = (0.05 \ 0.1 \ 0 \ 0.85) \cdot \begin{pmatrix}
0.6 & 0.2 & 0.2 & 0 \\
0.3 & 0.4 & 0.3 & 0 \\
0.1 & 0.2 & 0 & 0.7 \\
0.05 & 0.1 & 0 & 0.85
\end{pmatrix} = \begin{pmatrix}
0.1025 & 0.135 & 0.04 & 0.7225
\end{pmatrix} $$

Finally, we can compute the expected profit for each of the three possible bets:

$$ E[W] = 0.1025 \cdot 3.5 = 0.35875 $$

$$ E[T] = 0.135 \cdot 4 = 0.54 $$

$$ E[L] = (0.04 + 0.7225) \cdot 1.5 = 1.14375. $$

Now, the best choice is to bet on a loss. Clearly, the addition of the state $LL$ worsens the situation for FCB.

## 2 Probability of Arrival

The proof is similar to the one about the expected hitting time $h_{ij}$ (see script). We express $f_{ij}$ as a condition probability that depends on the result of the first step in the Markov chain. Recall that the random variable $T_{ij}$ is the hitting time, that is, the number of steps from $i$ to $j$. We get $Pr[T_{ij} < \infty \ | \ X_1 = k] = Pr[T_{kj} < \infty] = f_{kj}$ for $k \neq j$ and $Pr[T_{ij} < \infty \ | \ X_1 = j] = 1$. We can
therefore write $f_{ij}$ as follows.

\[
f_{ij} = \Pr[T_{ij} < \infty] = \sum_{k \in S} \Pr[T_{ij} < \infty \mid X_1 = k] \cdot p_{ik}
\]

\[
= p_{ij} \cdot \Pr[T_{ij} < \infty \mid X_1 = j] + \sum_{k \neq j} \Pr[T_{ij} < \infty \mid X_1 = k] \cdot p_{ik}
\]

\[
= p_{ij} + \sum_{k \neq j} p_{ik} f_{kj}
\]

3 Basketball

a) This exercise is a good example to illustrate that most exercises allow several differing solutions.

**Variant A.** Let $X$ be a random variable for the number of shots scored by Mario and $X_i$ an indicator variable that the $i$-th shot scores. Then obviously $X = \sum_{i=1}^{n} X_i$ when $n$ is the number of shots performed. The probability that the $i$-th attempt scores is $p$ as given in the exercise. Hence, we can use linearity of expectation to obtain the expectation of $X$.

\[
E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = n \cdot p
\]

We want Mario to score $m$ times.

\[
E[X] = n \cdot p = m \iff n = \frac{m}{p}
\]

Hence, Mario needs $\frac{m}{p}$ attempts to score $m$ times. After these $\frac{m}{p}$ attempts, Mario has scored an expected $m$ hits and he has missed expected $\frac{m}{p} - m$ times. Hence, he does an expected $10(\frac{m}{p} - m)$ push-ups in the game.

**Variant B.** We define a random variable $X$ that counts the number of attempts until we miss for the first time. $X$ is distributed as follows:

\[
\Pr[X = 1] = (1 - p)
\]

\[
\Pr[X = 2] = p(1 - p)
\]

\[
\vdots
\]

\[
\Pr[X = i] = p^{i-1}(1 - p)
\]

We say that $X$ is geometrically distributed with parameter $(1 - p)$ or write $X \sim \text{Geom}(1 - p)$. The expected value of a geometrically distributed random variable with parameter $\alpha$ is $\frac{1}{\alpha}$.

\[
E[X] = \frac{1}{1 - p}
\]

Again, due to the linearity of the expected value we may think of the game as Mario scoring $E[X] - 1$ hits, missing once, scoring the next $E[X] - 1$ hits, missing again, and so forth until he scored a total of $m$ hits. The question of how often Mario misses now translates to the question of how many series of $E[X] - 1$ successful attempts he needs in order to score $m$ times, and we get $10 \cdot \frac{m}{p} - m$ push-ups in expectation.

**Variant C (Markov Chain).** The following Markov chain models Mario’s game.

\[
\begin{array}{cccccc}
q_0 & p & q_1 & p & q_2 & p & \cdots & p & q_m \\
1 - p & 1 - p & 1 - p & 1 - p & & & & & 1 - p
\end{array}
\]
In a state $q_i$ Mario has scored $i$ hits. To learn the expected number of attempts until Mario has scored $m$ hits we can simply compute the hitting time $h_{0m}$ from $q_0$ to $q_m$.

$$h_{0m} = 1 + \sum_{k \neq m} p_{0k} h_{km} = 1 + p_{00} h_{0m} + p_{01} h_{1m}$$

$$h_{0m} = \frac{1 + p_{01} h_{1m}}{1 - p_{00}} = \frac{1 + ph_{1m}}{p} = \frac{1}{p} + h_{1m}$$

$$h_{1m} = 1 + p_{11} h_{1m} + p_{12} h_{2m} \iff h_{1m} = \frac{1}{p} + h_{2m}$$

$$h_{0m} = \frac{1}{p} + h_{1m} = \frac{2}{p} + h_{2m} = \ldots = \frac{m}{p} + h_{mm} = \frac{m}{p}$$

By subtracting the $m$ successful attempts, we get an expected $\frac{m}{p} - m$ misses and hence Mario does $10(\frac{m}{p} - m)$ push-ups in expectation.

b) Each sequence of (at most $m$) throws where Luigi tries to score $m$ times is called a round. A non-successful round is followed by push-ups.

Let $X$ be a random variable for the number of rounds that Luigi has to perform until he hits $m$ shots straight. The probability that Luigi scores $m$ consecutive shots is $p^m$. Observe that $X$ is geometrically distributed with parameter $p^m$ (cf. Exercise 3a variant B) and hence

$$E[X] = \frac{1}{p^m}$$

In the last round (which was successful), Luigi does not do any push-ups, hence we expect him to do $10 \cdot \left(\frac{1}{p^m} - 1\right)$ push-ups.

c) The following Markov chain models Trudy’s game.

In state $q_i$ Trudy has scored $i$ hits in a row, in $q_M$ she has missed once, in $q_G$ she has missed twice in a row and gives up.

(i) We determine the probability $f_{S3}$ of reaching the accepting state $q_3$ from the start state $q_S$.

$$f_{S3} = p \cdot f_{13} + (1 - p) \cdot f_{M3}$$

$$f_{13} = p \cdot f_{23} + (1 - p) \cdot f_{M3}$$

$$f_{23} = p + (1 - p) \cdot f_{M3}$$

$$f_{M3} = p \cdot f_{13}$$
\[ f_{13} = p^2 + (1-p)p^2 \cdot f_{13} + (1-p)p \cdot f_{13} = \frac{p^2}{1+p^3-p} = 0.4 \]

\[ f_{S3} = p \cdot \frac{p^2}{1+p^3-p} + (1-p)p \cdot \frac{p^2}{1+p^3-p} = \frac{2p^3-p^4}{1+p^3-p} = 0.3 \]

The probability that Trudy scores 3 times in a row is 0.3. The probability that she gives up is 0.7. This is because \( q_3 \) and \( q_G \) are the only absorbing states, i.e., all other states have probability mass of 0 in the steady state.

(ii) To get the number of push-ups we define a random variable \( Z \) that counts how often the system passes state \( q_M \) before either ending up in state \( q_3 \) or in state \( q_G \). E.g., the probability \( \Pr[Z=1] \) of passing \( q_M \) exactly once equals the probability of getting from \( q_S \) to \( q_M \) without being absorbed by \( q_3 \) and then ending up directly in \( q_G \) or \( q_3 \), i.e. \( \Pr[Z=1] = P_{SM} \cdot (P_{MG} + P_{M3}) \) where \( P_{ij} \) is the probability of getting from \( q_i \) to \( q_j \) without passing \( q_M \) on the way. \( Z \) has the following probability distribution:

\[
\begin{align*}
\Pr[Z=0] &= 1 - P_{SM} \\
\Pr[Z=1] &= P_{SM} \cdot (P_{MG} + P_{M3}) \\
\Pr[Z=2] &= P_{SM} \cdot P_{MM} \cdot (P_{MG} + P_{M3}) \\
\Pr[Z=3] &= P_{SM} \cdot P_{MM}^2 \cdot (P_{MG} + P_{M3}) \\
\vdots \\
\Pr[Z=i] &= P_{SM} \cdot P_{MM}^{i-1} \cdot (P_{MG} + P_{M3})
\end{align*}
\]

The probability of passing \( q_M \) exactly \( i \) times equals the probability of getting from \( q_S \) to \( q_M \) and from \( q_M \) to \( q_M \) again \( i-1 \) times and then ending up directly in \( q_G \) or \( q_3 \). As the Markov chain is not too complicated we can compute the needed \( P_{ij} \) rather easily and get \( P_{SM} = 1-p^3, P_{MM} = p-p^3, P_{MG} = 1-p, \) and \( P_{M3} = p^3 \).

The expected number of misses is

\[
\mathbb{E}[Z] = \sum_{i=1}^{\infty} i \cdot \Pr[Z = i] \\
= \sum_{i=1}^{\infty} i \cdot P_{SM} \cdot P_{MM}^{i-1} \cdot (P_{MG} + P_{M3}) \\
= P_{SM} \cdot (P_{MG} + P_{M3}) \cdot \sum_{i=1}^{\infty} i \cdot P_{MM}^{i-1} \\
= \frac{P_{SM} \cdot (P_{MG} + P_{M3})}{(1-P_{MM})^2} \\
= \frac{(1-p^3) \cdot (1-p+p^3)}{(1-p+p^3)^2} = \frac{1-p^3}{1-p+p^3} \\
= \frac{1 - \frac{1}{5}}{1 - \frac{3}{5} + \frac{1}{5}} = \frac{7}{5} = 1.4.
\]

Hence, Trudy does 14 push-ups in expectation.
Variant. We already know that Trudy gives up with a probability 0.7. Each time Trudy is in $q_M$ she gets to $q_G$ with probability $1 - p$. Hence it must hold that $E[Z] \cdot (1 - p) = 0.7$. This yields for the expected number of push-ups

$$10 \cdot E[Z] = 10 \cdot \frac{0.7}{1 - p} = 10 \cdot 2 \cdot 0.7 = 14.$$