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Swiss Federal Institute of Technology Zurich

Distributed Computing

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Discrete Event Systems

Solution to Exercise Sheet 9

1 Gloriabar

a) You might be tempted to model this situation by a queue with a bounded number of states, because the maximal number of persons in the line is bounded by 540. However, the situation can also be modeled by an infinite M/M/1 queue without losing too much accuracy; the parameter ρ will not be too large, such that the probability to reach the state in which 540 persons are standing in the queue at once is extremely small anyway. The modeling by an infinite M/M/1 queue conveniently allows us to apply Little's Law (slides 4/98 ff.) and therefore, we can use the formulae for the response and waiting time from slide 4/102:

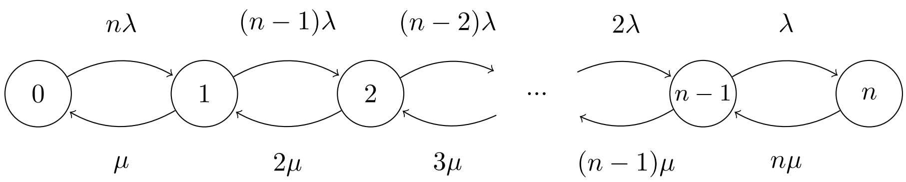
We have an arrival rate of $\lambda = 540/(90 \cdot 60) = 1/10$ (persons per second), and $\mu = 1/9$ (persons per second). Thus $\rho = \lambda/\mu = 9/10$. Applying Little's Law and the mentioned formulae yields: The expected waiting time is $W = \rho/(\mu - \lambda) = 81$ seconds; the expected time until the student has paid for her menu is given by $T = 1/(\mu - \lambda) = 90$ seconds.

b) We use the formula for the expected number of jobs in the queue from slide 4/102 and obtain queue length of $N_Q = \rho^2/(1-\rho) = 8.1$.

c) We require that $T = 1/(\mu - 0.1) = 90/2$. Thus, $\mu = 11/90$, i.e., instead of 9 secs, the service time should be $90/11 \approx 8.2$ secs.

2 Beachvolleyball

a) We know that the minimum of i independent and exponentially distributed (with parameter λ) random variables is an exponentially distributed random variable with parameter $i\lambda$. Thus, we have the following birth-death-process:



b) Let π_i be the probability of state i in the equilibrium. From slide 4/110, we know that

$$\pi_i = \pi_0 \cdot \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}}$$

and thus

$$\pi_i = \pi_0 \cdot \frac{\lambda_0 \cdot \lambda_1 \cdot \dots \cdot \lambda_{i-1}}{\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_i}.$$

Applying this formula to our process yields

$$\pi_i = \pi_0 \cdot \frac{n(n-1) \cdot \dots \cdot (n-i+1) \cdot \lambda^i}{1 \cdot 2 \cdot \dots \cdot i \cdot \mu^i} = \pi_0 \cdot \binom{n}{i} \cdot \rho^i \tag{1}$$

where $\rho := \frac{\lambda}{\mu}$. We know that the sum of all probabilities equals 1, so we have

$$\sum_{i=0}^{n} \pi_i = \pi_0 \sum_{i=0}^{n} \binom{n}{i} \rho^i = 1$$

Using the given formula for the binomial series

$$\sum_{i=0}^{n} \binom{n}{i} x^i = (1+x)^n$$

we obtain

$$\pi_0 (1+\rho)^n = 1 \quad .$$

Finally, we obtain

$$\pi_i = \frac{\binom{n}{i}\rho^i}{(1+\rho)^n} \quad .$$

c) (i) It is $\rho = 3/9 = 1/3$. We calculate the probability that there are less than two fit players:

$$\begin{aligned}
\pi_0 + \pi_1 &= \frac{1}{(1+\rho)^n} \cdot \left(1 + \binom{n}{1} \cdot \rho^1\right) \\
&= \left(\frac{3}{4}\right)^5 \cdot \left(1 + \frac{5}{3}\right) \\
&= \frac{3^5}{2^{10}} \cdot \frac{8}{3} \\
&= \frac{3^4}{2^7} \approx 0.63
\end{aligned}$$

Thus, the Disco team cannot participate in the tournament with probability 0.63.

(ii) Now, $\rho = 2/4 = 0.5$. Again, we calculate $\pi_0 + \pi_1$:

$$\begin{aligned}
\pi_0 + \pi_1 &= \frac{1}{(1+\rho)^n} \cdot \left(1 + \binom{n}{1} \cdot \rho^1\right) \\
&= \frac{1}{1.5^5} \cdot (1 + 0.5 \cdot 5) \\
&= \frac{2^5 \cdot 3.5}{3^5} \approx 0.46
\end{aligned}$$

Hence, the probability that the Disco team cannot participate is 0.46!

(iii) In general, if $\rho \geq 1$, an M/M/1 queue might grow infinitely and therefore doesn't have a stationary distribution. This cannot happen in this birth-and-death process, though, because there is only a bounded number of states. Hence, the process has a stationary distribution even for $\rho \geq 1$.

3 Theory of Ice Cream Vending

The situation can be described by a classic M/M/2 system. According to slide 4/113, there is an equilibrium iff

$$\rho = \lambda/(2\mu) < 1 \ .$$

For the stationary distribution, it holds that

$$\begin{aligned}
\pi_0 &= \frac{1}{\left(\sum_{k=0}^{m-1} \frac{(\rho m)^k}{k!}\right) + \frac{(\rho m)^m}{m!(1-\rho)}} \\
&= \frac{1}{\frac{(2\rho)^0}{0!} + \frac{(2\rho)^1}{1!} + \frac{(2\rho)^2}{2!(1-\rho)}} \\
&= \frac{1}{1 + 2\rho + \frac{4\rho^2}{(2(1-\rho))}} \\
&= \frac{1}{1 + 2\rho + \frac{4\rho^2}{2(1-\rho)}} \\
&= \frac{1}{\frac{2(1-\rho)+4\rho(1-\rho)+4\rho^2}{2(1-\rho)}} \\
&= \frac{2(1-\rho)}{2 - 2\rho + 4\rho - 4\rho^2 + 4\rho^2} \\
&= \frac{2(1-\rho)}{2 + 2\rho} \\
&= \frac{1-\rho}{1+\rho} \ .
\end{aligned}$$