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Distributed Computing

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Discrete Event Systems

Solution to Exercise Sheet 13

1 Network Calculus

Recall that α fulfills the arrival curve property if

$$\forall t\ \forall s : R(t) - R(s) \leq \alpha(t-s)\ ,$$

and β is a service curve if

$$\forall t\ \exists s : R^*(t) - R(s) \geq \beta(t-s)\ .$$

a) If $R \leq R \otimes \alpha$, then by definition for all t:

$$R(t) \leq (R \otimes \alpha)(t) = \inf_u \{R(t-u) + \alpha(u)\}.$$

As this inequality holds for the infimum over all u, it will also hold for *any* u, especially also for $u = t - s$ with *arbitrary* s. With this, we get

$$\begin{aligned} & R(t) && \leq R(t-(t-s)) + \alpha(t-s) \\ \Longrightarrow\quad & R(t) - R(s) && \leq \alpha(t-s), \end{aligned}$$

which is what we had to show for the first property: The inequality holds for *all* s and t.

b) Similarly, with the definitions from the lecture and the exercise sheet we get for all t

$$R^*(t) \geq (R \otimes \beta)(t) = \inf_u \{R(t-u) + \beta(u)\}.$$

Let u_0 be the u realizing the infimum, and let $s := t - u_0$, i.e. $u_0 = t - s$. Replacing u by u_0 and removing the infimum yields

$$\begin{aligned} & R^*(t) && \geq R(t-(t-s)) + \beta(t-s) \\ \Longrightarrow\quad & R^*(t) - R(s) && \geq \beta(t-s). \end{aligned}$$

Thus, for all t there exists some $s := t - u_0$ fulfilling the inequality, which is exactly what we had to show.

2 Power-Down Mechanisms

As mentioned in the hint, we only focus on a single idle period because if we know that our algorithm is c-competitive for any idle period, we also know that it is c-competitive for the complete busy sequence.

a) Analogously to the 2-competitive ski-rental online algorithm, we consider an algorithm Alg that powers down after D time units. To see that Alg is 2-competitive, we distinguish two cases for the length of the current idle period T:

- $T < D$: The energy consumed by both algorithms is $c_{\text{ALG}} = c_{\text{OPT}} = T$, hence the competitive ratio is $c = T/T = 1$.
- $T \geq D$: We have $c_{\text{ALG}} = D + D$ since ALG waits D time units and then powers-down and $c_{\text{OPT}} = D$ because OPT powers down immediately. Hence we get

$$c = \frac{2D}{D} = 2 \ .$$

b) Let ALG be any *deterministic* power down algorithm. Then the time t_{ALG} after which it powers down in an idle period is known in advance. The "worst" idle period ends immediately after ALG has powered down, that is we have $T = t_{\text{ALG}} + \varepsilon$. Again, we distinguish two cases with respect to the time t_{ALG} when ALG powers down.

- $t_{\text{ALG}} < D$: We have $c_{\text{ALG}} = t_{\text{ALG}} + D$ and $c_{\text{OPT}} = t_{\text{ALG}} + \varepsilon$, hence

$$c = \frac{t_{\text{ALG}} + D}{t_{\text{ALG}} + \varepsilon} = 1 + \frac{D - \varepsilon}{t_{\text{ALG}} + \varepsilon} > 2 \quad \text{for } \varepsilon \to 0$$

since $t_{\text{ALG}} < D$.

- $t_{\text{ALG}} \geq D$: We have $c_{\text{ALG}} = t_{\text{ALG}} + D$ again and $c_{\text{OPT}} = D$, hence

$$c = \frac{t_{\text{ALG}} + D}{D} = 1 + \frac{t_{\text{ALG}}}{D} \geq 2 \quad \text{for } \varepsilon \to 0$$

since $t_{\text{ALG}} \geq D$.

Hence, ALG cannot be better than 2-competitive.

c) Let ALG be a randomised algorithm that powers down at time $\frac{2}{3}D$ with probability $\frac{1}{2}$ and at time D otherwise. Let C_{ALG} be a random variable for the cost incurred by the algorithm. We again consider an arbitrary idle period of length T. We distinguish three cases:

- $T < \frac{2}{3}D$: The energy consumption of both algorithms is $c_{\text{ALG}} = c_{\text{OPT}} = T$, hence $c = T/T = 1 < 2$.
- $\frac{2}{3}D \leq T < D$: The expected energy consumption of ALG is

$$\mathbf{E}[C_{\text{ALG}}] = \frac{1}{2}\left(\frac{2}{3}D + D\right) + \frac{1}{2}T = \frac{5}{6}D + \frac{1}{2}T$$

and further $c_{\text{OPT}} = T$. Hence we get

$$c = \frac{\frac{5}{6}D + \frac{1}{2}T}{T} = \frac{1}{2} + \frac{5}{6} \cdot \frac{D}{T} \leq \frac{1}{2} + \frac{5}{6} \cdot \frac{D}{\frac{2}{3}D} = \frac{1}{2} + \frac{5}{4} = \frac{7}{4} < 2 \ .$$

- $T \geq D$: We have for the expected energy consumption of ALG

$$\mathbf{E}[C_{\text{ALG}}] = \frac{1}{2}\left(\frac{2}{3}D + D\right) + \frac{1}{2}(D + D) = \frac{5}{6}D + D = \frac{11}{6}D$$

and further $c_{\text{OPT}} = D$. Hence we get

$$c = \frac{\frac{11}{6}D}{D} = \frac{11}{6} < 2 \ .$$

Hence, the randomised algorithm is $\frac{11}{6}$-competitive which is better than any deterministic algorithm.

Note: This result, however, is not optimal yet. The best randomised algorithm uses a continuous probability distribution for the shutdown time and thereby achieves a competitive ratio of $e/(e-1) \approx 1.58$.