Eidgenössische Technische Hochschule Zürich
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Computer Engineering and Networks Laboratory
Computer Engineering Group (TEC)

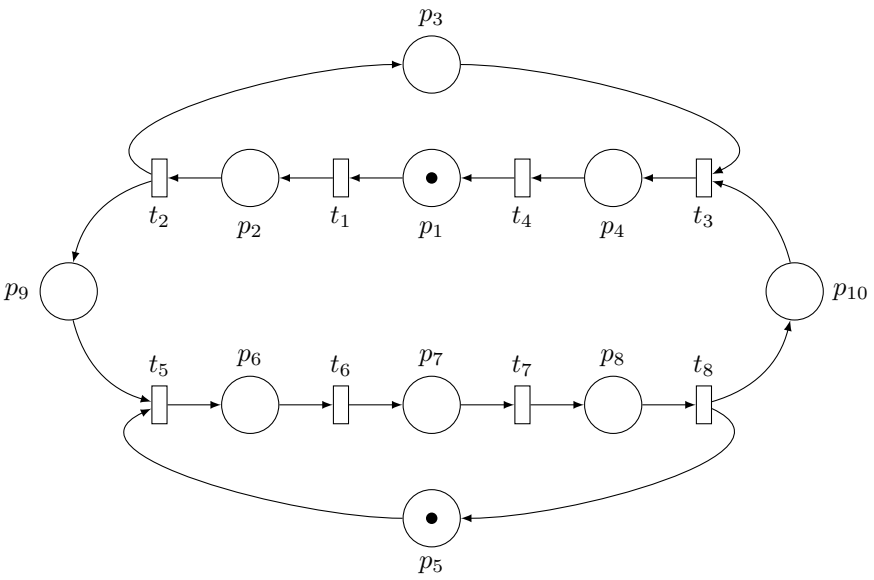
HS 2015 Prof. L. Thiele / R. Jacob

Discrete Event Systems

Solution to Exercise Sheet 13

1 Structural Properties of Petri Nets and Token Game

Given is the following Petri net N_1:



a) What are the Pre and Post sets of transitions t_5 and t_8 and of place p_3?

b) Which transitions are enabled after t_1 and t_2 fired?

c) What is the total number of tokens in N_1 before and after t_2 fired?

d) Play the token game for N_1 and construct the reachability graph.
Hint: You may denote the states in such a way that the index indicates the places that hold a token in this state, for example $\vec{s}_0 = (1,0,0,0,1,0,0,0,0,0) \triangleq s_{1,5}$.

a) The Pre and Post sets of a transition are defined as follows:

- Pre set: $\bullet t := \{p \mid (p,t) \in F\}$
- Post set: $t\bullet := \{p \mid (t,p) \in F\}$,

where F is the flow set, i.e., the set of place/transition and transition/place arcs. The Pre and Post sets of a place are defined analogously.

For the Petri net N_1 we obtain the following sets:

$$\begin{aligned} {\bullet t_5} &= \{p_5, p_9\}, & t_5{\bullet} &= \{p_6\} \\ {\bullet t_8} &= \{p_8\}, & t_8{\bullet} &= \{p_{10}, p_5\} \\ {\bullet p_3} &= \{t_2\}, & p_3{\bullet} &= \{t_3\} \end{aligned}$$

b) A transition is enabled if all places in its Pre set contain enough tokens. In the case of N_1, which has only unweighted edges, one token per place suffices. When t_2 fires, it consumes one token out of each place in the Pre set of t_2 and produces one token on each place in the Post set of t_2. Hence, the firing of t_2 produces one token on place p_3 and p_9 each, the one on p_2 is consumed. After this, t_5 is enabled because both p_9 and p_5 hold one token. However, t_3 is not enabled because p_3 contains a token but p_{10} does not.

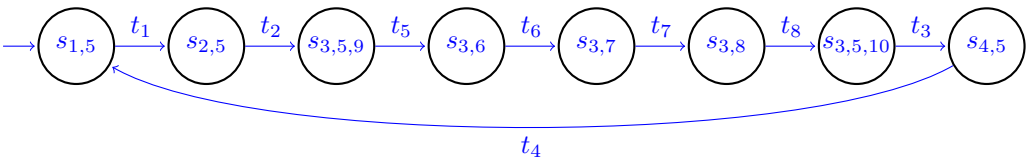
c) Before t_2 fires there are two tokens in N_1, one on p_2 and p_5 each. Directly afterwards, there are tokens on places p_3, p_9 and p_5, hence 3 tokens in total.

d) A token traverses the upper cycle until t_2 fires. Then one token remains on p_3 and waits, and another one is produced in p_9, which enables transition t_5. When t_5 consumes the tokens on p_9 and p_5 and produces a token on p_6, this one traverses the lower cycle until t_8 is enabled and fired. One token now remains on p_5 and waits, another is in p_{10} and enables t_3, because there is another token on p_3. Then one token traverses the upper cycle again until t_2 is enabled, and so on. Hence, this Petri net models two alternating processes.

This Petri net is clearly bounded, thus we can construct its reachability tree. Usually the states of Petri nets are denoted by vectors such that the i-th position in the vector indicates the number of tokens on place p_i of the Petri net, i.e., the marking of the graph. So, for example, the starting state $\vec{s}_0$ of N_1, in which the places p_1 and p_5 hold one token each, is denoted by $\vec{s}_0 = (1,0,0,0,1,0,0,0,0,0)$.

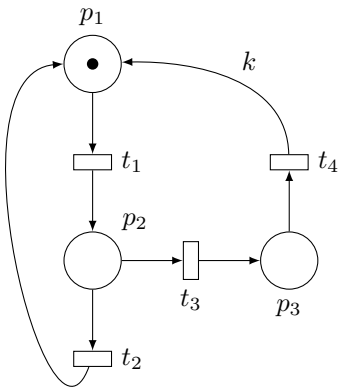
For better legibility we denote the states in such a way that the index contains the places that hold a token in this state, for example $\vec{s}_0 = (1,0,0,0,1,0,0,0,0,0) \stackrel{\triangle}{=} s_{1,5}$.

Then the reachability graph can also be written as,



2 Basic Properties of Petri Nets

Given is the following Petri net N_2:



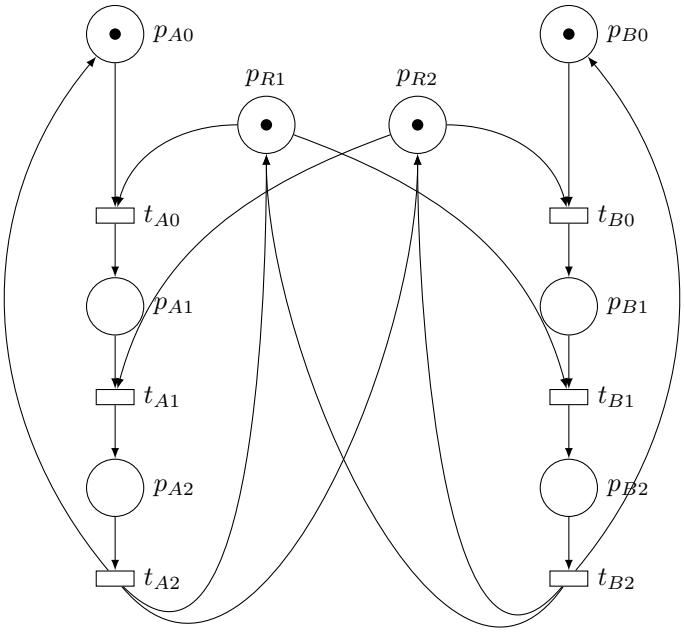
Explain the terms *boundedness* and *deadlock-freeness* using this example, i.e. for which values of $k \in \mathbb{N}$ is the Petri net N_2 bounded/unbounded and not deadlock-free?

A Petri net is k-bounded, if there is no fire sequence that makes the number of tokens in one place grow larger than k. It is obvious that Petri net N_2 is 1-bounded if $k \leq 1$. This holds because in the initial state there is only one token in the net, and in the case $k \leq 1$ no transition increases the number of tokens in N_2. If $k \geq 2$, the number of tokens in p_1 can grow infinitely large by repeatedly firing t_1, t_3 and t_4. So, the Petri net N_2 is unbounded for $k \geq 2$.

A Petri net is deadlock free if no fire sequence leads to a state in which no transition is enabled. If $k = 0$, N_2 is not deadlock-free. The fire sequence t_1, t_3, t_4 causes the only existing token to be consumed and hence, there is no enabled transition any more. For $k \geq 1$, however, no deadlock can occur.

3 Identifying a deadlock

The following Petri net N_3 describes two linear processes ($P_{A0/A1/A2}$ and $P_{B0/B1/B2}$) sharing resources R_1 and R_2.



In the following, use $M = (P_{A0}, P_{A1}, P_{A2}, P_{R1}, P_{R2}, P_{B0}, P_{B1}, P_{B2})$ as marking vector and $T = (t_{A0}, t_{A1}, t_{A2}, t_{B0}, t_{B1}, t_{B2})$ as firing vector.

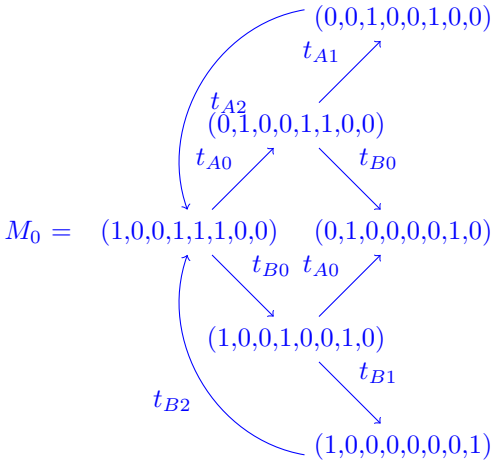
a) Determine the reachability graph of this net for the given initial marking. Explicit one or several firing sequences leading to a blocking marking (i.e., to a deadlock). What is this blocking marking?

b) Write down the upstream (W^-) and downstream (W^+) incidence matrices and deduce the incidence matrix A. Use it to compute the marking obtained in the deadlock state (i.e., by firing the blocking sequence) from the previous question.

c) Using the upstream incidence matrix W^-, how can you **prove** that this previous state is a deadlock?

d) Suggest a modification to this Petri net which allows the two linear processes P_A and P_B to run as intended in the first place.

a) There are an infinite number of blocking sequence: any number of cycles $t_{A0}t_{A1}t_{A2}$ and/or $t_{B0}t_{B1}t_{B2}$ terminated by either $t_{A0}t_{B0}$ or $t_{B0}t_{A0}$. It can be read directly from the marking graph below:



b) From the Petri net structure, we get:

$$W^+ = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, W^- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$A = W^+ - W^- = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

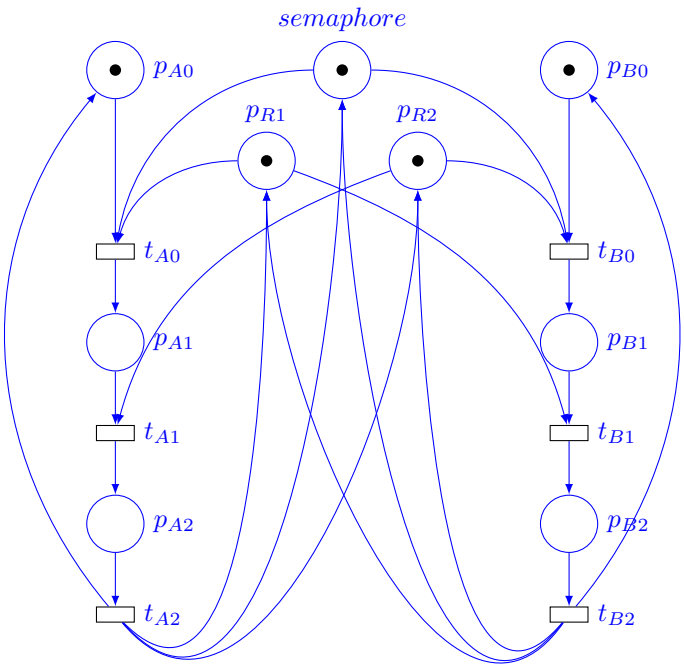
Consider the basic blocking sequence $t_{A0}t_{B0}$. It entails:

$$M_{deadlock} = M_0 + A \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

As expected, we find again the blocking marking from the reachability graph of the previous question.

c) A deadlock state is a state at which no transition in enabled. Hence, one can use the upstream transition matrix W^- to assess whether or not a marking is blocking. It is the case if and only if the marking vector does not **cover** (i.e., is strictly bigger than...) any of W^- column. Otherwise, it implies the transition associated to this column is enabled, hence this marking is not blocking.

d) In order to avoid that deadlock, we need to forbid both process to run concurrently. This can be solved easily using a semaphore, as illustrated thereafter:



4 From mutual exclusion to starvation

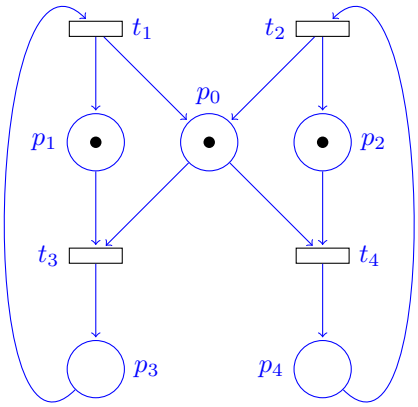
Your task is to model a system as a Petri net in which two processes want to access a common exclusive resource is a similar fashion as in Exercise 3. This means that the two processes have to exclude each other mutually from the concurrent access to the resource (e.g. a critical program section). More precisely:

1. A process executes its program.
2. In order to enter the critical section, a given mutex variable must be 0.
3. If this is the case, the process sets the mutex to 1 and executes its critical section.
4. When done, it resets the mutex to 0 and enters an uncritical section.
5. Then the procedure starts all over again.

a) Propose a Petri net representing the desired behavior.
Hint: use 5 places and 4 transitions.

b) In this setting, it may happen that a process starves the other. That means one process always uses the resource and the other never enters the critical section. Correct this in such a way that each process cannot get the resource more than twice in a row. This may yield that only one process can start running from the initial marking.

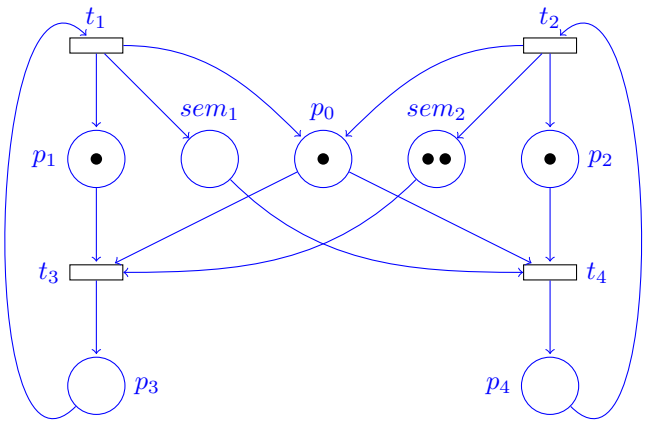
a) For each process we introduce two places (p_1, p_2, p_3 and p_4) representing the process within the normal program execution (p_1, p_2) as well as in the critical section (p_3, p_4). For each process, we have a token indicating which section of the program is currently executed. Additionally, we introduce a place p_0 representing the mutex variable. If the mutex variable is 0, then we have a token at p_0. We have to make sure that a process can only enter its critical section if there is a token at the mutex place. The resulting Petri net looks as follows.



Assume that initially, both processes are in an non-critical section (in the Petri net, this is denoted by a token in place p_1 and p_2 respectively). A process can only enter its critical section (p_3/p_4) if there is a token at p_0. In this case, the token is consumed when entering the critical section. A new mutex token at p_0 is not created until the process leaves its critical section. Hence, both processes exclude each other mutually from the concurrent access to the critical section.

This is a classical benefit of Petri nets over other DES models. It models very efficiently the sharing of resources, the concurrency of processes, and so on...

b) In order to avoid starvation of either of the process, one option is to count the number of execution the each of them, or more precisely the difference between them. Assume that at initial state, none has been previously run. According to the specification, we can allow one to the process (say A) to run twice by creating a "counter-resource" with 2 tokens at initial state. Running the process A consume one of these tokens and a new token will be generated in this place on completion of process B. Doing that symmetrically (on each process) yields the number of executions of each process together...
Lots of rather obscure explanations. Want the net?



5 Reachability Analysis for Petri Nets

In the lecture we presented an algorithm to perform a reachability analysis on Petri nets.

a) Why is it not possible with a reachability algorithm to determine *in general*, whether a given state in a Petri net is reachable or not?

b) Consider the Petri net N_2 from exercise 2. Is the state $s = (p_1 = 101, p_2 = 99, p_3 = 4)$ reachable from the initial state $s_0 = (1, 0, 0)$ if $k = 2$? Prove your answer.

Hint: Start with the necessary condition presented in the lecture for the reachability of a state in a weighted Petri net, then eventually explain whether or not the marking is reachable.

a) Petri nets may possess infinite reachability graphs, e.g. N_2 with $k \geq 2$. If the state in question is actually reachable in such a Petri net, the reachability algorithm will eventually terminate. If it is not reachable, the algorithm will never be able to determine this with absolute certainty (cf. halting problem).

b) We determine the incidence matrix of the Petri net as explained in the lecture.

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

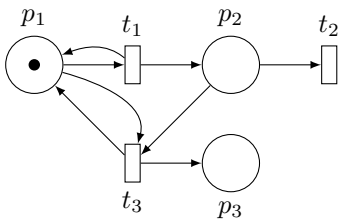
We are interested in whether the state $\vec{s} = (101, 99, 4)$ is reachable from the initial state $\vec{s_0} = (1, 0, 0)$. If the equation system $\mathbf{A} \cdot \vec{f} = \vec{s} - \vec{s_0}$ has no solution, we know that the state $\vec{s}$ is not reachable from s_0. "Unfortunately",

$$\begin{pmatrix} -1 & 1 & 0 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 99 \\ 4 \end{pmatrix}$$

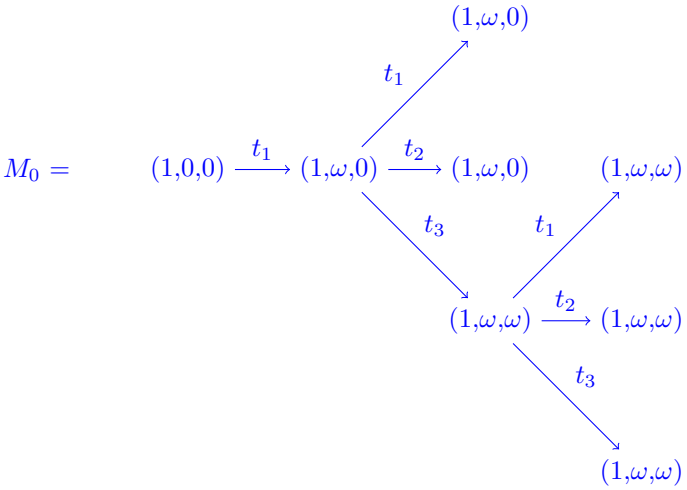
is satisfiable. To show that $\vec{s}$ is reachable from $\vec{s_0}$, we have to give a firing sequence through which we get from $\vec{s_0}$ to $\vec{s}$. From the last equation of the above equation system, we know that $f_3 = f_4 + 4$. Hence, in the desired firing sequence, f_3 is fired four times more than f_4. However, $\vec{f}$ does not tell us about the firing order. Considering the Petri net, we can see that – starting from $\vec{s_0}$ – the number of tokens in p_1 increases by one after firing t_1, t_3, and t_4 in this order. Repeating this for 203 times yields the state $(204, 0, 0)$. Firing t_1 for 103 times followed by firing t_3 for four times finally yields state $\vec{s}$.

6 Coverability tree and graph

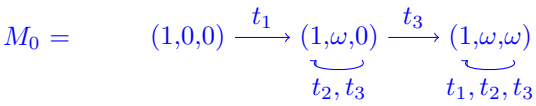
Given is the following Petri net N_6, compute its coverability tree and coverability graph. Deduce which are the unbounded places of this net given the initial marking.



Following the procedure from the lecture note, we can construct the following coverability tree:



One can merge the equivalent node and obtain the coverability graph:



It follows that for this net with initial marking $(1, 0, 0)$, places p_2 and p_3 are unbounded.