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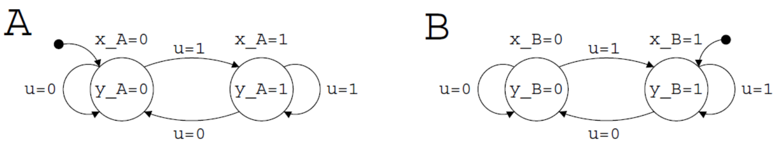
Discrete Event Systems

Solution to Exercise Sheet 10

1 Comparison of Finite Automata

Here are two simple finite automata:



For each, we have a one bit encoding for the states (x_A and x_B), one binary output (y_A and y_B), and one common binary input (u). We want to verify whether or not these two automata are equivalent. This can be done through the following steps:

a) Express the characteristic function of the transition relation for both automaton, $\psi_r(x, x', u)$.

b) Express the joint transition function, ψ_f.
Reminder: $\psi_f(x_A, x'_A, x_B, x'_B) = (\exists u : \psi_A(x_A, x'_A, u) \cdot \psi_B(x_B, x'_B, u))$.

c) Express the characteristic function of the reachable states, $\psi_X(x_A, x_B)$.

d) Express the characteristic function of the reachable output, $\psi_Y(y_A, y_B)$.

e) Are the two automata equivalent? **Hint:** Evaluate, for example, $\psi_Y(0, 1)$.

a) $\psi_A(x_A, x'_A, u) = \overline{x_A}\,\overline{x'_A}\,\overline{u} + \overline{x_A}x'_A u + x_A x'_A u + x_A \overline{x'_A}\,\overline{u}$
$\psi_B(x_B, x'_B, u) = \overline{x_B}\,\overline{x'_B}\,\overline{u} + \overline{x_B}x'_B u + x_B x'_B u + x_B \overline{x'_B}\,\overline{u}$

b)
$$\begin{aligned}\psi_f(x_A, x'_A, x_B, x'_B) &= (\overline{x_A}x'_A + x_A x'_A) \cdot (\overline{x_B}x'_B + x_B x'_B) + \\ &\quad (\overline{x_A}\,\overline{x'_A} + x_A \overline{x'_A}) \cdot (\overline{x_B}\,\overline{x'_B} + x_B \overline{x'_B}) \\ &= \overline{x_A}x'_A\overline{x_B}x'_B + \overline{x_A}x'_A x_B x'_B + x_A x'_A \overline{x_B} x'_B + x_A x'_A x_B x'_B + \\ &\quad \overline{x_A}\,\overline{x'_A}\,\overline{x_B}\,\overline{x'_B} + \overline{x_A}\,\overline{x'_A} x_B \overline{x'_B} + x_A \overline{x'_A}\,\overline{x_B}\,\overline{x'_B} + x_A \overline{x'_A} x_B \overline{x'_B}\end{aligned}$$

c) Computation of the reachable states is performed incrementally. Starts with the initial state of the system $\psi_{X_0}(x_A, x_B) = \overline{x_A} x_B$ and then add the successors until reaching a fix-point,

$$\begin{aligned}\psi_{X_1} &= \psi_{X_0} + (\exists (x'_A, x'_B) : \psi_{X_0}(x_A, x_B) \cdot \psi_f(x_A, x'_A, x_B, x'_B)) \\ &= \overline{x_A} x_B + \overline{x_A}\,\overline{x_B} + x_A x_B \\ \psi_{X_2} &= \overline{x_A} x_B + \overline{x_A}\,\overline{x_B} + x_A x_B = \psi_{X_1} \quad \rightarrow \text{the fix-point is reached!}\end{aligned}$$

$\Rightarrow$ $\boxed{\psi_X = \overline{x_A} x_B + \overline{x_A}\,\overline{x_B} + x_A x_B}$

d) Here you first need to express the output function of each automaton, that is the feasible combinations of states and outputs,

$\psi_{g_A} = \overline{x_A y_A} + x_A y_A$ and $\psi_{g_B} = \overline{x_B y_B} + x_B y_B$

Then the reachable outputs are the combination of the reachable states and the outputs functions, that is,

$$\psi_Y(y_A, y_B) = (\exists (x_A, x_B) : \psi_X \cdot \psi_{g_A} \cdot \psi_{g_B})$$
$$= y_A y_B + \overline{y_A y_B} + \overline{y_A} y_B$$

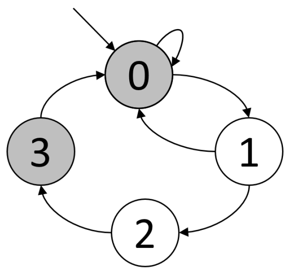
e) From the reachable output function, we see that these automata are not equivalent. Indeed, there exists a reachable output admissible $(\psi_Y((y_A, y_B) = (0,1)) = 1)$ for which $y_A \neq y_B$.

Another way of saying looking at it: $\psi_Y \cdot (y_A \neq y_B) \neq 0$,

where $(y_A \neq y_B) = \overline{y_A} y_B + y_A \overline{y_B}$.

2 Temporal Logic

a) We consider the following automaton. The property a is true on the colored states (0 and 3).



For each of the following CTL formula, list all the states for which it holds true.

(i) EF a

(ii) EG a

(iii) EX AX a

(iv) EF (a `AND` EX `NOT`(a))

(i) $Q = \{0, 1, 2, 3\}$

(ii) $Q = \{0, 3\}$

(iii) (AX a) holds for $\{2, 3\}$, thus $Q = \{1, 2\}$

(iv) (a `AND` EX `NOT`(a)) is true for states where a is true and there exists a direct successor for which it is not. Only state 0 satisfy this (from it you can transition to 1, where a does not hold). Moreover, state 0 is reachable for all states in this automaton ("from all states there exists a path going through 0 at some point") Hence $Q = \{0, 1, 2, 3\}$

b) Given the transition function $\psi_f(x, x')$ and the characteristic function $\psi_Z(x)$ for a set Z, write a small pseudo-code which returns the characteristic function of $\psi_{\text{AF } Z}(x)$. It can be expressed as symbolic boolean functions, like $\overline{x_A} x'_A \overline{x_B} x'_B + \overline{x_A} x'_A x_B x'_B$.

Hint: To do this, simply use the classic boolean operators `AND`, `OR`, `NOT` and $!=$. You can also use the operator `PRE`(Q, f), which returns the predecessor of the set Q by the transition function f. That is,

$$\texttt{PRE}(Q, f) = \{q' : \exists x, \psi_f(q', q) \cdot \psi_Q(q) = 1\}$$

Hint: It can be useful to reformulate AFZ as another CTL formula.

Here, the trick is to remember that AF $Z \equiv$ NOT(EG NOT(Z)). Hence, one can compute the function for EG NOT(Z) quite easily (following the procedure given in the lecture) and take the negation in the end. A possible pseudo-code doing this is the following,

Require: ψ_Z, ψ_f ▷ Equivalence in term of sets:

```
current = NOT(ψ_Z);                          ▷ X_0
next = current AND ψ_PRE(current,f);         ▷ X_1 = X_0 ∩ Pre(X_0, f)
while next != current do                     ▷ X_i != X_{i-1}
    current = next;
    next = current AND ψ_PRE(current,f);     ▷ X_i = X_{i-1} ∩ Pre(X_{i-1}, f)
end while                                    ▷ X_f |= EG NOT(Z)
return ψ_AF Z = NOT(current);                ▷ overline(X_f) |= AF Z = NOT(EG NOT(Z))
```