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Networked Systems Group (NSG)

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based on Prof. R. Wattenhofer's material

# Discrete Event Systems

## Exercise Sheet 3

## 1 Pumping Lemma [Exam]

**The Pumping Lemma in a Nutshell**

Given a language $L$, assume for contradiction that $L$ is regular and has the pumping length $p$. Construct a suitable word $w \in L$ with $|w| \geq p$ ("there *exists* $w \in L$") and show that for *all* divisions of $w$ into three parts, $w = xyz$, with $|x| \geq 0$, $|y| \geq 1$, and $|xy| \leq p$, there *exists* a pumping exponent $i \geq 0$ such that $w' = xy^iz \notin L$. If this is the case, $L$ is not regular.

Language $L_1$ can be shown to be non-regular using the pumping lemma. Assume for contradiction that $L_1$ is regular and let $p$ be the corresponding pumping length. Choose $w$ to be the word $0110^p1^p$. Because $w$ is an element of $L_1$ and has length more than $p$, the pumping lemma guarantees that $w$ can be split into three parts, $w = xyz$, where $|xy| \leq p$ and for any $i \geq 0$, we have $xy^iz \in L_1$. In order to obtain the contradiction, we must prove that for every possible partition into three parts $w = xyz$ where $|xy| \leq p$, the word $w$ cannot be pumped. We therefore consider the various cases.

**a)** If $y$ starts anywhere within the first three symbols (i.e. 011) of $w$, deleting $y$ (pumping with $i = 0$) creates a word with an illegal prefix (e.g. $1\,0^p\,1^p$ for $y = 01$).

**b)** If $y$ consists of only 0s from the second block, the word $w' = xy^2z$ has more 0s than 1s in the last $|w'| - 3$ symbols and hence $c \neq d$.

Note that $y$ cannot contain 1s from the second block because of the requirement $|xy| \leq p$.

We have shown that for all possible divisions of $w$ into three parts, the pumped word is not in $L_1$. Therefore, $L_1$ cannot be regular and we have a contradiction.

**Be Careful!**

The argumentation above is based on the closure properties of regular languages and only works in the direction presented. That is, for an operator $\diamond \in \{\cup, \cap, \bullet\}$, we have:

If $L_1$ and $L_2$ are regular, then $L = L_1 \diamond L_2$ is also regular.

If either $L_1$ or $L_2$ or both are non-regular, we cannot deduce the non-regularity of $L$ or vice-versa. Moreover, $L$ being regular does not imply that $L_1$ and $L_2$ are regular as well. This may sound counter-intuitive which is why we give examples for the three operators.

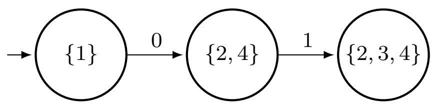
- $L = L_1 \cup L_2$: Let $L_1$ be any non-regular language and $L_2$ its complement. Then $L = \Sigma^*$ is regular.

- $L = L_1 \cap L_2$: Let $L_1$ be any non-regular language and $L_2$ its complement. Then $L = \emptyset$ is regular.
- $L = L_1 \bullet L_2$: Let $L_1 = \{a^*\}$ (a regular language) and $L_2 = \{a^p \mid p \text{ is prime}\}$ (a non-regular language) then $L = \{aaa^*\}$ is regular.

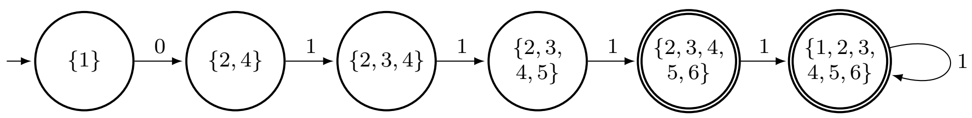
Hence, to prove that a language $L_x$ is non-regular, you assume it to be regular for contradiction. Then you combine it with a *regular* language $L_r$ to obtain a language $L = L_x \diamond L_r$. If $L$ is non-regular, $L_x$ could not have been regular either.

## 2 Deterministic Finite Automata [Exam]

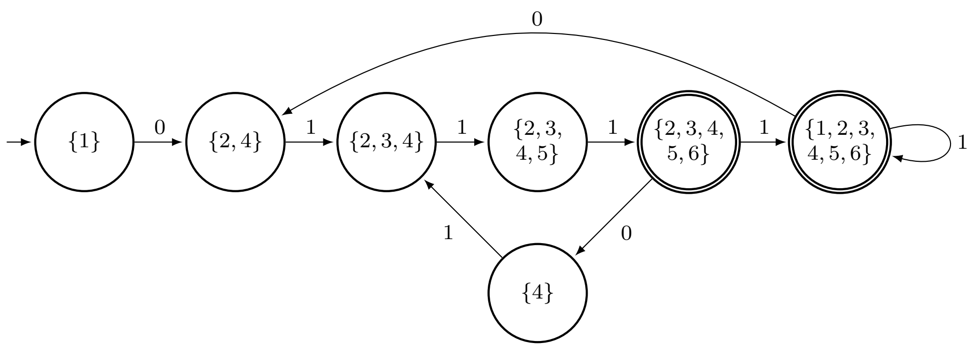
We could use the systematic transformation scheme presented in the lecture (slide 1/75). Considering the large number of states, however, this will easily lead to an explosion of states in the derandomized automaton. Hence, we build the deterministic finite automaton in a step-wise manner, only creating those states that are actually required: Initially, the automaton requires a 0. Subsequently, only a 1 is accepted. Including the various transitions, this 1 can lead to three different states, namely states 2, 3, and 4.



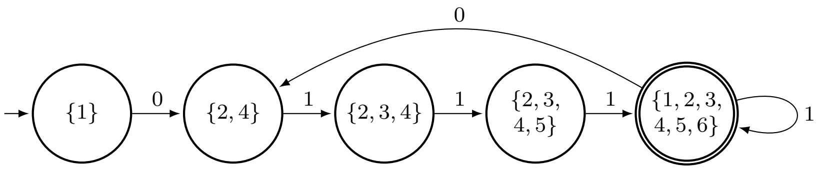

In any of the states 2, 3, and 4, only a 1 is accepted. Assume that the automaton is currently in state 2, this 1 can lead to states $\{2, 3, 4\}$ when including all $\varepsilon$-transitions. When in state 3, the 1 leads to states $\{2, 3, 4, 5\}$ and finally, when being in state 4, the reachable states given a 1 are $\{2, 3, 4\}$. Hence, a 1 leads from state $\{2, 3, 4\}$ to state $\{2, 3, 4, 5\}$. Repeating the same process for state $\{2, 3, 4, 5\}$, we can see that, again, only a 1 is accepted, which leads to state $\{2, 3, 4, 5, 6\}$. Because the state 6 in the original NFA was an accepting state, $\{2, 3, 4, 5, 6\}$ is also accepting in the DFA. From state $\{2, 3, 4, 5, 6\}$, an additional 1 will lead to another accepting state $\{1, 2, 3, 4, 5, 6\}$. And from this state, any subsequent 1 returns to state $\{1, 2, 3, 4, 5, 6\}$ as well.



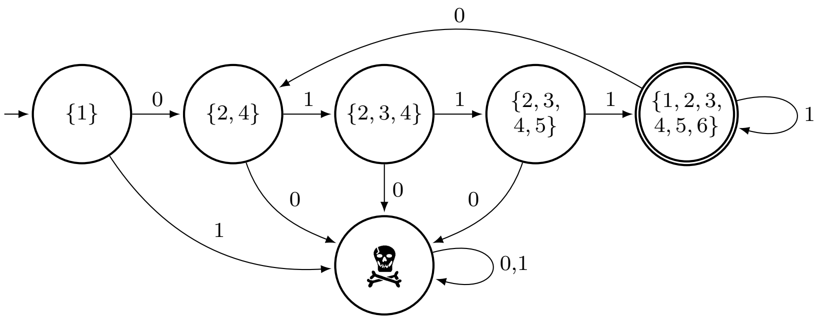

What happens if a 0 occurs in the input? This is feasible only when the deterministic state includes either state 1 or state 6. In state $\{2, 3, 4, 5, 6\}$, a 0 necessarily leads to state $\{4\}$, whereas in state $\{1, 2, 3, 4, 5, 6\}$ a 0 leads to state $\{2, 4\}$. In both of these states, the only acceptable input symbol is a 1 and leads to the state $\{2, 3, 4\}$. Hence, the deterministic finite automaton looks like this:



It can easily be seen, that first the states $\{4\}$, $\{2,4\}$ and then the states $\{2,3,4,5,6\}$, $\{1,2,3,4,5,6\}$ can be merged and hence, the automaton can be reduced to the one shown in the next figure.

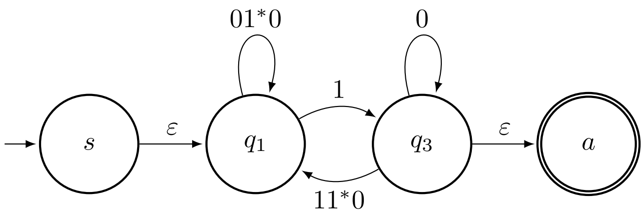



This is not a DFA yet, because the crash state is still missing. The final deterministic automaton looks like this:

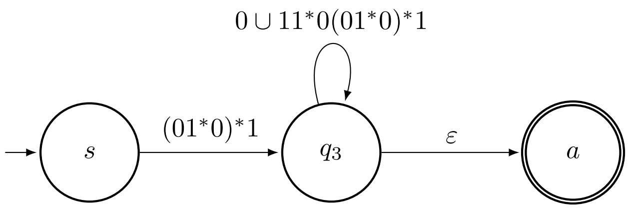



# 3 Transforming Automata [Exam]

The regular expression can be obtained from the finite automaton using the transformation presented in the script on slide 1/85. After ripping out state $q_2$, the corresponding GNFA looks like this:




After also removing state $q_1$, the GNFA looks as follows.




Eliminating the last state $q_3$ yields the final solution, which is $(01^*0)^*1(0 \cup 11^*0(01^*0)^*1)^*$.

*Note:* Ripping out the interior states in a different order yields a distinct yet equivalent regular expression. The order $q_3, q_2, q_1$, for example, results in $((0 \cup 10^*1)1^*0)^*10^*$.

# 4 Regular and Context-Free Languages

**a)** Sometimes, even simple grammars can produce tricky languages. We can interpret the 1s and 2s of the second production rule as opening and closing brackets. Hence, $L(G)$ consists of all correct bracket terms where at least one 0 must be in each bracket.

Choose $w = 1^p02^p \in L(G)$. Let $w = xyz$ with $|xy| \leq p$ and $|y| \geq 1$ (pumping lemma). Because of $|xy| \leq p$, $xy$ can only consist of 1s. According to the pumping lemma, we should have $xy^iz \in L$ for all $i \geq 0$. However, by choosing $i = 0$ we delete at least one 1 and get a word $w' = 1^{p-|y|}02^p$ with $|y| \geq 1$. $w'$ is not in $L$ since it has fewer 1s than 2s. This means that $w$ is not pumpable and hence, $L(G)$ is not regular.

**b)** Since *every* regular language is also context-free, we can choose an arbitrary regular language. For example, we can choose the language $L = \{0^n1, n \geq 1\}$ which is clearly regular. A context-free grammar for this language uses only the production $S \to 0S \mid 1$.

# 5 Context-Free Grammars

**a)** An example for a grammar $G$ producing the language $L_1$ is $G = (V, \Sigma, R, S)$ with

$$\begin{aligned} V &= \{X, A\}, \\ \Sigma &= \{0, 1\}, \\ R &= \begin{Bmatrix} X \to XAX \mid A, \\ A \to 0 \mid 1 \end{Bmatrix}, \\ S &= X \end{aligned}$$

*Note*: The language is regular!

**b)** A rather natural grammar generating $L_2$ uses the following productions:

$$\begin{aligned} S \to\ & A1A \\ A \to\ & A1 \mid 1A \mid A01 \mid 0A1 \mid 01A \mid A10 \mid 1A0 \mid 10A \mid \varepsilon \end{aligned}$$

Another slightly more complicated solution yielding simpler productions looks as follows:

$$\begin{aligned} S \to\ & A1A \\ A \to\ & AA \mid 1A0 \mid 0A1 \mid 1 \mid \varepsilon \end{aligned}$$

The idea of both grammars is to first ensure that there is at least one 1 more and then have a production that generates all possible strings with the same number of 0s and 1s or further 1s at arbitrary places.

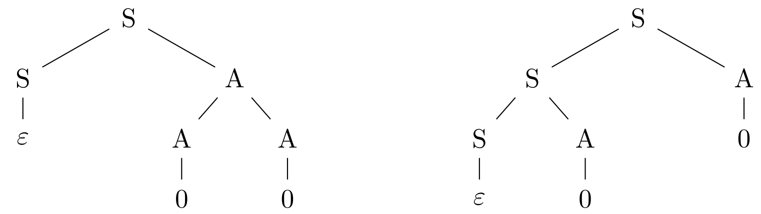
# 6 Ambiguity

**a)** $\varepsilon, 0, 00, (), (0), 0(), ()0, 000$

**b)** It is ambiguous, because the word 00 has two different leftmost derivations.

$$\begin{aligned} S &\to SA \\ &\to A \\ &\to AA \\ &\to 0A \\ &\to 00 \end{aligned} \qquad\qquad \begin{aligned} S &\to SA \\ &\to SAA \\ &\to AA \\ &\to 0A \\ &\to 00 \end{aligned}$$

It can also be seen by taking a look at these two derivation trees that both belong to the word 00:




Because the two derivation trees are structurewise different, the word 00 can be derived ambiguously from $G$.

**Ambiguity of Grammars**

*Definition:* A string $s$ is derived *ambiguously* in a context-free grammar $G$ if it has two or more different leftmost/rightmost derivations (or two structurewise different derivation trees). Grammar $G$ is *ambiguous* if it generates some string ambiguously.

A *leftmost/rightmost* derivation replaces in every step the leftmost/rightmost variable.

*Example:* The grammar with the productions '$S \to S \cdot S \mid S + S \mid a$' is ambiguous since the string $s = a \cdot a + a$ has two different leftmost derivations.

$$\begin{aligned} S &\to S \cdot S & \qquad S &\to S + S \\ &\to a \cdot S & &\to S \cdot S + S \\ &\to a \cdot S + S & &\to a \cdot S + S \\ &\to a \cdot a + S & &\to a \cdot a + S \\ &\to a \cdot a + a & &\to a \cdot a + a \end{aligned}$$

Intuitively, the derivation on the left corresponds to the arithmetic expression $a \cdot (a+a)$ because we first derive a product and then substitute one factor by a sum while the derivation on the right corresponds to $(a \cdot a) + a$ because we first have a sum and then substitute one summand by a product.

The productions of an equivalent non-ambiguous grammar are $A \to S + a \mid S \cdot a \mid a$.