Eidgenössische Technische Hochschule Zürich
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DYNAMO

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based on Prof. Dr. Lothar Thiele's material

# Discrete Event Systems

## Exercise Sheet 11

A complex system often has its components designed independently and assembled into the overall system. For example, a car manufacturer assembles different parts that are produced by specialized suppliers. The combined behavior of the assembled components results in the overall system behavior.

Some combined behaviors are allowed (e.g., you can listen to the radio while driving), whereas some are forbidden (e.g., the engine must not start if the gas tank is open). Therefore, the interaction between components needs to be controlled, which is the goal of the *system specification and verification.*

In this exercise session, we look at these two steps. In the first exercise, we use set representations and characteristic functions to express system properties. In the second and third exercises, we represent the system design and specification using Boolean expressions, and we use *binary decision diagrams* (BDDs) as a means to verify that the design satisfies the specification.

# 1 Set Representation

## 1.1 Set Operations and Characteristic Functions

In the early design phase of a system, it is common to identify some states as *faulty*, or *error* states. One goal is to ensure that, the system never enters such states. We define the following sets of states:

- $X$: the set of *all* states.
- $N$: the set of *normal* states.
- $E$: the set of *error* states.
- $O$: the set of states where *there is a memory overflow.*

**Definition 1** *For a given state $x$, a binary encoding function $\sigma(x)$,and a set of states $Q$, the* characteristic function *of $Q$ (denoted as $\psi_Q(\cdot)$) is defined as following*

$$\psi_Q(\sigma(x)) = \begin{cases} 0, & x \notin Q \\ 1, & x \in Q \end{cases} \quad . \tag{1}$$

In each of the following sub-questions, we are interested in describing the property of the given sets of states using set operations (i.e., using set operators $\cup$, $\cap$, and $\overline{\bullet}$ and set symbols $X$, $N$, $E$, and $O$) and characteristic functions.

**a)** What is $\psi_X$, the characteristic function of *the set of all states*?

**b)** *"Each state (in the set of all states) is either a normal or an error state or both"*. Express this property in terms of sets and characteristic functions.

**c)** *"If a state (in the set of all states) is in the set of overflow states, it is not a normal state"*. Express this property in terms of sets and characteristic functions.

**d)** Describe $Q_1$, the set of *error states* which are not an overflow, in terms of sets and characteristic functions.

**Definition 2** *A state $s$ is in the set of states $O \Rightarrow E$ (reads $O$ implies $E$) if $s$ in $O$, then it is in $E$.*

**e)** Describe $Q_2 = (O \Rightarrow E)$ in terms of characteristic functions.

### 1.2 Write Specifications in Boolean Encoding and Compose Them

Consider a sensor network with 3 *sensor nodes*, a *bus* and a *sink* (i.e., a node where data is collected). The network can be in the *normal mode* or *bootstrapping mode.* A node can be in the *sleep mode* or in the *awake mode.* In order to save energy, nodes are put in sleep mode whenever possible.

Below is a set of constraints described in plain texts, and a set of Boolean variables that encode the system states. Your task consists in (1) expressing each of the constraints individually, and (2) combining them to express the overall system specification.

The list of constraints:

- **C1**: When *one or more* nodes are using the bus, the sink must be awake to receive data.
- **C2**: *No more than one* node can use the bus at the same time.
- **C3**: When the network is in bootstrapping mode, then the sink must be awake, and the nodes cannot use the bus.

The encoding:

- $x_1 = 1$: Node 1 is using the bus.
- $x_2 = 1$: Node 2 is using the bus.
- $x_3 = 1$: Node 3 is using the bus.
- $x_s = 1$: The sink is in awake mode.
- $x_b = 1$: The network is in bootstrapping mode.

**a)** Express the specification of **C1**, **C2** and **C3** using the given Boolean encoding.

**b)** What is the specification of the desired behavior? **Hint:** All constraints should be satisfied.

## 2 Sets of States and State Transitions

*Adapted from the written exam of HS 2023.*

Figure 1 depicts a system of 3 light bulbs; there is 1 push button under each light bulb. The left side of Figure 1 describes the *initial state* of the system—light bulb 1 is in the *on* state, and light bulbs 2 and 3 are in the *off* state. A state of this system is represented using a 3-bit–binary encoding with state bits $x$, $y$, $z$. If light bulb 1 is in the *on* state, $x := 1$, otherwise, $x := 0$; the encodings for $y$ and $z$ are defined analogously. For instance, the initial state *on*, *off*, *off* is encoded as $(x, y, z) := (1, 0, 0)$.

In each state, we move to the next state by pushing *one and exactly one button*: whenever a button is pushed, the light bulb is directly above it and the adjacent light bulbs are toggled. For example, suppose that we are in the current state $(x, y, z) := (1, 0, 0)$; if we push button 3, the states of the light bulbs of the next state will be *on, on, on*, i.e., $(x', y', z') := (1, 1, 1)$.

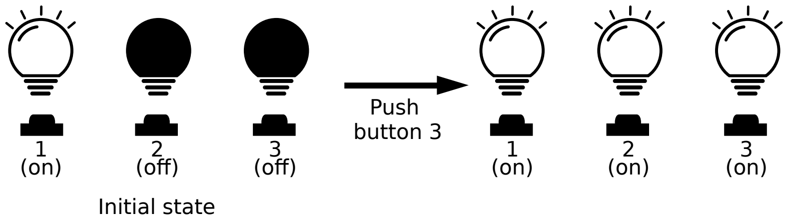



Figure 1: A system with 3 light bulbs.

Consider the state machine described above. The state space of the state machine is denoted as $S$ and its set of state transitions as $T : S \times S$. The *characteristic function* of $T$ is denoted as $\psi_T(x, y, z, x', y', z')$, where $x, y, z$ are the state variables of the current state and $x', y', z'$ are the state variables for the next state.

Write the corresponding characteristic functions of the following set of states or set of translation relations:

**a)** Determine the characteristic function $\psi_1(x, y, z)$ of the set of states that satisfy the following constraint: *if light bulb 1 is in the* on *state, then exactly one of the other light bulbs, 2 or 3, must be in the* on *state.* Use only Boolean operators.

**b)** Consider the set of initial states $Q_0 := \{(1, 0, 0)\}$ with the characteristic function $\psi_{Q_0}$. Determine the characteristic function $\psi_2(x, y, z, x', y', z')$ of the set of state transitions starting from any state in the set of initial states, i.e., $\psi_{Q_0}(x, y, z) \cdot \psi_T(x, y, z, x', y', z')$.

# 3 Binary Decision Diagrams

For a *reduced ordered binary decision diagram* (ROBDD) representing a Boolean function with $N$ binary variables, we denote the variable order by $\Pi :\ x_1 \prec x_2 \prec \ldots \prec x_n$ , where $x_1$ is the highest variable of the tree, $x_2$ the second highest, and so on. The notation $x_i \prec x_j$ is used to indicate that we evaluate $x_i$ before $x_j$. An ordering $\Pi_1$ is said to be better than $\Pi_2$ for an ROBDD $G$, if $G$ contains fewer decision nodes when using $\Pi_1$ rather than $\Pi_2$ (after merging equivalent nodes).

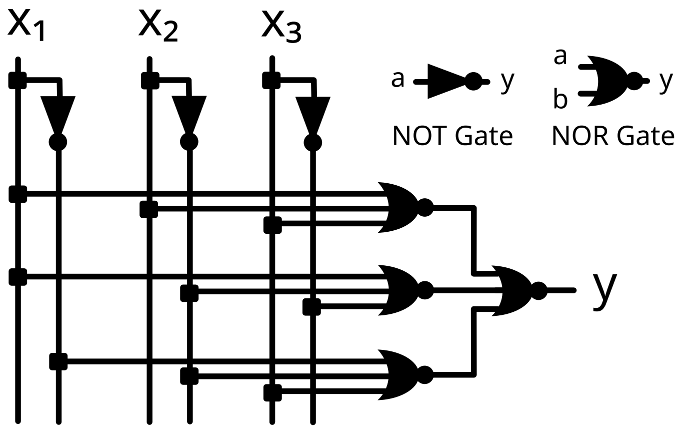
In the following, use the following notation to represent ROBDDs: use a solid arc (———) to indicate that the parent node evaluates to 1, and use a dashed arc (− − − −) otherwise. **Do not** use color to differentiate the type of nodes/edges (not allowed to use multiple colors in the exam).

## 3.1 Verifying the Equivalence of Combinational Circuits Using BDDs

You are in the process of designing a processing architecture. Your specification analysis results in the following desired function to implement:

$$F_1 := x_1\overline{x_2} + x_1x_3 + \overline{x_2}x_3 + \overline{x_1}x_2\overline{x_3}. \tag{2}$$

For practical reason, you only allowed to use **NOT** gates and **NOR** gates to implement your circuit. Considering this constraint, your synthesis tool returns the following schematic:




Your team leader is quite old-fashioned and does trust not these new fancy software so much (or maybe he/she is just testing you!). He/She asks that you verify the schematic circuit does indeed implement the same function as $F_1$. This can be done efficiently using ROBDDs.

**a)** Express the function $F_2$ realized by the circuit.

**b)** Draw and compare the ROBDDs of $F_1$ and $F_2$ using the ordering of variables $\Pi : x_1 \prec x_2 \prec x_3$. Do they implement the same behavior?

## 3.2 ROBDDs with Respect to Different Orderings

**a)** Consider a Boolean function $G(x_1, x_2, y_1, y_2) := (x_1 \leftrightarrow y_1) \cdot (x_2 \leftrightarrow y_2)$, where $(x \leftrightarrow y)$ denotes $x \cdot y + \overline{x} \cdot \overline{y}$. Write the Boole-Shannon decomposition of $G$ with respect to $\Pi : x_1 \prec x_2 \prec y_1 \prec y_2$.

**b)** Consider the same ordering of variables $\Pi$, draw the corresponding ROBDD for $G$.

**c)** Consider a new ordering $\Pi' : x_1 \prec y_1 \prec x_2 \prec y_2$. Use it to reconstruct the ROBDD of $G$. Is $\Pi'$ a better ordering than $\Pi$ for $G$?