Principles of Distributed Computing

Exercise 13

1 Flow labeling schemes

In this exercise, we focus on flow labeling schemes. Let $G = \langle V, E, w \rangle$ be a weighted undirected graph where, for every edge $e \in E$, the weight $w(e)$ is integral and represents the capacity of the edge. For two vertices $u, v \in V$, the flow between them (in either direction), denoted $\text{flow}(u, v)$, can be defined as follows. Denote by $G'$ the multigraph obtained by replacing each edge $e$ in $G$ with $w(e)$ parallel edges of capacity 1. A set of paths $P$ in $G'$ is edge-disjoint if each edge (with capacity 1) appears in no more than one path $p \in P$. Let $P_{u,v}$ be the collection of all sets $P$ of edge-disjoint paths in $G'$ between $u$ and $v$. Then $\text{flow}(u, v) = \max_{P \in P_{u,v}} |P|$.

Consider the family $G(n, \hat{\omega})$ of undirected weighted connected $n$-vertex graphs with maximum integral capacity $\hat{\omega}$. We will find flow labeling schemes for this family. Given a graph $G = \langle V, E, w \rangle$ in this family and an integer $1 \leq k$, define the relation:

$$R_k = \{(x, y) \mid x, y \in V, \text{flow}(x, y) \geq k\}.$$

**Question 1** Show that for every $k \geq 1$, the relation $R_k$ induces a collection of equivalence classes on $V$, $C_k = \{C_{k1}, \ldots, C_{km}\}$, such that $C_i \cap C_j = \emptyset$ (if $i \neq j$) and $\bigcup_i C_i = V$. What is the relationship between $C_k$ and $C_{k+1}$?

According to the solution of Question 1, given $G$, one can construct a tree $T_G$ corresponding to its equivalence relations. The $k$’th level of $T$ corresponds to the relation $R_k$. The tree is truncated at a node once the equivalence class associated with it is a singleton. For every vertex $v \in V$, denote by $t(v)$ the leaf in $T_G$ associated with the singleton set $\{v\}$.

For two nodes $x, y$ in a tree $T$ rooted at $r$, we define the separation level of $x$ and $y$, denoted $\text{SepLevel}_T(x, y)$, as the depth of $z = \text{lca}(x, y)$, the least common ancestor of $x$ and $y$. I.e., $\text{SepLevel}_T(x, y) = \text{dist}_T(z, r)$, the distance from $z$ to the root.

**Question 2**

a) Show that if there exists a labeling scheme for distance in trees with labeling size $L(\text{dist}, T)$, then there is a labeling scheme for separation level with labeling size $L(\text{SepLevel}, T) \leq L(\text{dist}, T) + \lceil \log m \rceil$ where $m$ is the number of nodes in the tree.

b) Recall there is an $O(\log^2 m)$ labeling scheme for distance in unweighted trees of size $m$. Show that $L(\text{flow}, G(n, \hat{\omega})) = O(\log^2 (n\hat{\omega}))$.

**Question 3** Assume there is an $O(\log^2 m + \log \omega \log m)$ labeling scheme for weighted distance in integer-weighted trees of size $m$ with max. weight size $\omega$.

Find a more careful design of the tree $T_G$ which can improve the bound on the label size to $L(\text{flow}, G(n, \hat{\omega})) = O(\log n \log \hat{\omega} + \log^2 n)$. Hint: Consider the nodes of degree 2 in $T_G$.

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1 As a convention, $\text{flow}(x, x) = \infty$. 