Principles of Distributed Computing
Exercise 14

1 Flow labeling schemes

In this exercise, we focus on flow labeling schemes. Let \( G = (V, E, w) \) be a weighted undirected graph where, for every edge \( e \in E \), the weight \( w(e) \) is integral and represents the capacity of the edge. For two vertices \( u, v \in V \), the flow between them (in either direction), denoted \( \text{flow}(u, v) \), can be defined as follows. Denote by \( G' \) the multigraph obtained by replacing each edge \( e \) in \( G \) with \( w(e) \) parallel edges of capacity 1. A set of paths \( P \) in \( G' \) is edge-disjoint if each edge (with capacity 1) appears in no more than one path \( p \in P \). Let \( P_{u,v} \) be the collection of all sets \( P \) of edge-disjoint paths in \( G' \) between \( u \) and \( v \). Then \( \text{flow}(u, v) = \max_{P \in P_{u,v}} \{|P|\} \).

Consider the family \( G(n, \hat{\omega}) \) of undirected weighted connected \( n \)-vertex graphs with maximum integral capacity \( \hat{\omega} \). We will find flow labeling schemes for this family. Given a graph \( G = (V, E, w) \) in this family and an integer \( 1 \leq k \), define the relation:

\[
R_k = \{ (x, y) \mid x, y \in V, \text{flow}(x, y) \geq k \}.
\]

Question 1 Show that for every \( k \geq 1 \), the relation \( R_k \) induces a collection of equivalence classes on \( V \), \( C_k = \{ C_k^1, \ldots, C_k^m \} \), such that \( C_k^i \cap C_k^j = \emptyset \) (if \( i \neq j \)) and \( \bigcup_i C_k^i = V \). What is the relationship between \( C_k \) and \( C_{k+1} \)?

According to the solution of Question 1, given \( G \), one can construct a tree \( T_G \) corresponding to its equivalence relations. The \( k' \)th level of \( T \) corresponds to the relation \( R_k \). The tree is truncated at a node once the equivalence class associated with it is a singleton. For every vertex \( v \in V \), denote by \( t(v) \) the leaf in \( T_G \) associated with the singleton set \{\( v \)\}.

For two nodes \( x, y \) in a tree \( T \) rooted at \( r \), we define the separation level of \( x \) and \( y \), denoted \( \text{SepLevel}_T(x, y) \), as the depth of \( z = \text{lca}(x, y) \), the least common ancestor of \( x \) and \( y \). I.e., \( \text{SepLevel}_T(x, y) = \text{dist}_T(z, r) \), the distance from \( z \) to the root.

Question 2

a) Show that if there exists a labeling scheme for distance in trees with labeling size \( L(\text{dist}, T) \), then there is a labeling scheme for separation level with labeling size \( L(\text{SepLevel}, T) \leq L(\text{dist}, T) + \lfloor \log m \rfloor \) where \( m \) is the number of nodes in the tree.

b) Recall there is an \( O(\log^2 m) \) labeling scheme for distance in unweighted trees of size \( m \). Show that \( L(\text{flow}, G(n, \hat{\omega})) = O(\log^2(n\hat{\omega})) \).

Question 3 Assume there is an \( O(\log^2 m + \log \omega \log m) \) labeling scheme for weighted distance in integer-weighted trees of size \( m \) with max. weight size \( \omega \).

Find a more careful design of the tree \( T_G \) which can improve the bound on the label size to \( L(\text{flow}, G(n, \hat{\omega})) = O(\log n \log \hat{\omega} + \log^2 n) \). Hint: Consider the nodes of degree 2 in \( T_G \).

\(^1\text{As a convention, flow}(x, x) = \infty.\)