Principles of Distributed Computing
Exercise 14: Sample Solution

1 Flow labeling schemes

Question 1  Check that \( R_k \) is reflexive, symmetric and transitive.

- reflexive: \( \text{flow}(x, x) = \infty \)
- symmetric: the graph is undirected, \( \text{flow}(x, y) = \text{flow}(y, x) \)
- transitive: consider a path \( p = (v_1, v_2, \ldots, v_m) \) from \( x \) to \( y \) in which \( v_1 = x \) and \( v_m = y \) and a path \( p' = (v'_1, v'_2, \ldots, v'_{m'}) \) from \( y \) to \( z \) in which \( v'_1 = y \) and \( v'_{m'} = z \). Let \( i \) be the largest subscript in \( p' \) such that \( v'_i \in p \). It is easy to check there is a path \( x \rightarrow v'_i \rightarrow z \) where \( x \rightarrow v'_i \) is a part of \( p \) and \( v'_i \rightarrow z \) is a part of \( p' \).

\( C_{k+1} \) is a refinement of \( C_k \).

Question 2

a) Add the depth of each vertex into the label. The depth of the tree is smaller than \( m \), so the added part is of size \( O(\log m) \). From the depth of two vertices and the distance between them, SepLevel can be computed.

b) Note that

\[
\text{flow}_G(v, w) = \text{SepLevel}_T(t(v), t(w)).
\]  

The depth of \( T_G \) cannot exceed \( n \hat{\omega} \) and every level at most has \( n \) nodes, hence the total number of nodes in \( T_G \) is \( O(n^2 \hat{\omega}) \).

Question 3  Cancel all nodes of degree 2 in \( T_G \), and add appropriate edge weights (\( \tilde{T}_G \)).

Now, define \( \text{SepLevel}_T(x, y) \) as the weighted depth of \( z = lca(x, y) \), i.e. its weighted distance from the root. Obtain the SepLevel labeling scheme for weighted trees in the same way as in question 2. For \( \tilde{n} \)-node trees with maximum weight \( \tilde{\omega} \), the labeling size is \( O(\log \tilde{n} \log \tilde{\omega} + \log^2 \tilde{n}) \).

Again, for two nodes \( x, y \) in \( G \), the weighted separation level of the leaves \( t(x) \) and \( t(y) \) associated with \( x \) and \( y \) in the tree \( T_G \) is related to the flow between the two vertices as in Eq. (1).

Finally, note that as \( \tilde{T}_G \) has exactly \( \tilde{n} \) leaves, and every non-leaf node in it has at least two children, the total number of nodes in \( \tilde{T}_G \) is \( \tilde{n} \leq 2n - 1 \). The maximum edge weight in \( \tilde{T}_G \) is \( \tilde{\omega} \leq n\hat{\omega} \). We end up with the label size of \( O(\log \tilde{n} \log \tilde{\omega} + \log^2 \tilde{n}) \).

For more details, see [1] (Section 2).
References