In this lecture, we discuss a rather general method for solving various local problems. The key concept in our discussion will be network decompositions first introduced by [ALGP89], also known as low-diameter graph decomposition [LS91].

0.1 Definition and Applications

Let us start with defining this concept.

**Definition 1.** (Weak Diameter Network Decomposition) Given a graph $G = (V, E)$, a $(C, D)$ weak diameter network decomposition of $G$ is a partition of $G$ into vertex-disjoint graphs $G_1, G_2, \ldots, G_C$ such that for each $i \in \{1, 2, \ldots, C\}$, we have the following property: the graph $G_i$ is made of a number of vertex-disjoint and mutually non-adjacent clusters $X_1, X_2, \ldots, X_\ell$, where each two vertices $v, u \in X_j$ have distance at most $D$ in graph $G$. We note that we do not bound the number $\ell$. We refer to each subgraph $G_i$ as one block of this network decomposition.

**Definition 2.** (Strong Diameter Network Decomposition) Given a graph $G = (V, E)$, a $(C, D)$ strong diameter network decomposition of $G$ is a partition of $G$ into vertex-disjoint graphs $G_1, G_2, \ldots, G_C$ such that for each $i \in \{1, 2, \ldots, C\}$, we have the following property: each connected component of $G_i$ has diameter at most $D$.

Notice that a strong diameter network decomposition is also a weak diameter network decomposition.

Network decompositions can be used to solve a wide range of local problems. To see the general method in a concrete manner, let us go back to our beloved $(\Delta + 1)$-coloring problem.

**Theorem 3.** Provided an $(C, D)$ weak-diameter network decomposition of a graph $G$, we can compute a $\Delta + 1$ coloring of $G$ in $O(CD)$ rounds.

**Proof.** We will color graphs $G_1, G_2, \ldots, G_C$ one by one, each time considering the coloring assigned to the previous subgraphs. Suppose that vertices of graphs graphs $G_1, G_2, \ldots, G_i$ are already colored using colors in $\{1, 2, \ldots, \Delta + 1\}$. We explain how to color $G_{i+1}$ in $O(D)$ rounds. Consider the clusters $X_1, X_2, \ldots, X_\ell$ of $G_{i+1}$ and notice their two properties: (1) they are mutually non-adjacent, (2) for each cluster $X_j$, its vertices are within distance $D$ of each other (where distances are according to the base graph $G$). For each cluster $X_j$, let node $v_j \in X_j$ who has the maximum identifier among nodes of $X_j$ be the leader of $X_j$. Notice that leaders of clusters $X_1, X_2, \ldots, X_\ell$ can be identified in $O(D)$ rounds (why?). Then, let $v_j$ aggregate the topology of the subgraph induced by $X_j$ as well as the colors assigned to nodes adjacent to $X_j$ in the previous graphs $G_1, G_2, \ldots, G_i$. This again can be done in $O(D)$ rounds, thanks to the fact that all the relevant information is within distance $D + 1$ of $v_j$. Once this information is gathered, node $v_j$ can compute a $(\Delta + 1)$-coloring for vertices of $X_j$, while taking into account the colors of neighboring nodes of previous graphs, using a simple greedy procedure. Then, node $v_j$ can report back these colors to nodes of $X_j$. This will happen for all the clusters $X_1, X_2, \ldots, X_\ell$ in parallel, thanks to the fact that they are non-adjacent and thus, their coloring choices does not interfere with each other.  

\[ \square \]
0.2 Randomized Algorithm for Network Decomposition

**Theorem 4.** There is a randomized LOCAL algorithm that computes a \((C, D)\) weak-diameter network decomposition of any \(n\)-node graph \(G\), for \(C = O(\log n)\) and \(D = O(\log n)\), in \(O(\log^2 n)\) rounds, with high probability\(^1\).

As we see in the exercises of this class, the two key parameters \(C\) and \(D\) are nearly optimal and one cannot improve them simultaneously and significantly.

**Network Decomposition Algorithm:** Suppose that we have already computed subgraphs \(G_1, \ldots, G_i\) so far. We now explain how to compute a subgraph \(G_{i+1} \subseteq G \setminus (\bigcup_{j=1}^{i} G_j)\), in \(O(\log n)\) rounds, which would satisfy the properties of one block of a weak diameter network decomposition.

Let each node \(v\) pick a random radius \(r_u\) from an geometric distribution with parameter \(\varepsilon\), for a desired (free parameter) constant \(\varepsilon \in (0, 1)\). That is, for each integer \(y \geq 1\), we have \(\Pr[r_u = y] = \varepsilon(1-\varepsilon)^{y-1}\). We will think of the vertices within distance \(r_u\) of \(u\) as the ball of node \(u\). Now for each node \(v\), let \(\text{Center}(v)\) be the node \(u^*\) among nodes \(u\) such that \(\text{dist}_G(u, v) \leq r_u\) that has the smallest identifier. The is, \(\text{Center}(v) = u^*\) is the smallest-identifier node whose ball contains \(v\). Define the clusters of \(G_i\) by letting all nodes with the same center define one cluster, and then discarding nodes who are at the boundary of their cluster. That is, any node \(v\) for which \(\text{dist}_G(v, u) = r_u\) where \(u = \text{Center}(v)\) remains unclustered.

There are two properties to prove: one that the clusters have low diameter, and second, that after \(C\) iterations, all nodes are clustered. In the following two lemmas, we argue that with high probability, each cluster has diameter \(O(\log n/\varepsilon)\) and after \(C = O(\log_1/\varepsilon n)\) iterations, all nodes are clustered.

**Lemma 5.** With high probability, the maximum cluster diameter is at most \(O(\log n/\varepsilon)\). Hence, this clustering can be computed in \(O(\log n/\varepsilon)\) rounds, with high probability.

**Proof.** The proof is simple and is left as an exercise.

**Lemma 6.** For each node \(v\), the probability that \(v\) is not clustered — that \(v\) is on the boundary of its supposed cluster and thus it gets discarded — is at most \(\varepsilon\).

**Proof.** Notice that

\[
\sum_{u \in V} \Pr[v \text{ is not clustered}] = \Pr[v \text{ is not clustered} | \text{Center}(v) = u] \cdot \Pr[\text{Center}(v) = u]
\]

For each vertex \(u\), let \(\text{before}(u)\) denote the set of all vertices whose identifier is less than that of \(u\). Define the following events

- \(\mathcal{E}_1 = (r_u = \text{dist}_G(v, u))\).
- \(\mathcal{E}_2 = (r_u \geq \text{dist}_G(v, u))\).
- \(\mathcal{E}_3 = (\forall u' \in \text{before}(u), r_{u'} < \text{dist}_G(v, u'))\).

\(^1\)Throughout, we will use the phrase with high probability to indicate that an event happens with probability at least \(1 - \frac{1}{n^c}\), for a desirably large but fixed constant \(c \geq 2\).
We have

\[
\begin{align*}
\Pr \ [v \ is \ not \ clustered \ | \ Center(v) = u] &= \Pr[\mathcal{E}_1 \cap \mathcal{E}_3 | \mathcal{E}_2 \cap \mathcal{E}_3] \\
&= \frac{\Pr[\mathcal{E}_1 \cap \mathcal{E}_3]}{\Pr[\mathcal{E}_2 \cap \mathcal{E}_3]} \\
&= \frac{\Pr[\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3]}{\Pr[\mathcal{E}_2 \cap \mathcal{E}_3]} \\
&= \frac{\Pr[\mathcal{E}_1 \cap \mathcal{E}_2]}{\Pr[\mathcal{E}_2 \cap \mathcal{E}_3]} \\
&= \frac{\Pr[\mathcal{E}_1]}{\Pr[\mathcal{E}_2]} = \varepsilon,
\end{align*}
\]

where in the penultimate equality, we used the property that the event \( \mathcal{E}_3 \) is independent of events \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \), and the last equality follows from the probability distribution function of the exponential distribution (recall that this is exactly the memoryless property of the exponential distribution). Hence, we can now go back and say that

\[
\begin{align*}
\Pr \ [v \ is \ not \ clustered ] &= \sum_{u \in V} \Pr[v \ is \ not \ clustered \ | \ Center(v) = u] \cdot \Pr[Center(v) = u] \\
&= \sum_{u \in V} \varepsilon \cdot \Pr[Center(v) = u] = \varepsilon.
\end{align*}
\]

**Corollary 7.** After \( C = O(\log_{1/\varepsilon} n) \) iterations, all nodes are clustered, with high probability.

**References**
