

The Locality of Maximal Matching

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standard synchronous message-passing model of distributed computing

• undirected graph $G = (V, E)$, n nodes, maximum degree Δ

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every problem is trivially solvable in $O(diameter)$ **rounds**

Easy centralized problems: greedy solutions.

Matching: set of non-incident edges

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Maximal: no edge can be added

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greedy property!

Centralized (Sequential) Algorithm

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 $\mathbb{E}[\#$ removed edges per round] $\geq c |E_i|$

 $\mathbb{E}[\text{Hremoved edges per round}] \geq c |E_i|$ $O(log n)$ rounds w.h.p.

Our Result

deterministic $O(log^2 \Delta \cdot log n)$ -round Maximal Matching

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improving over

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 $O(\log^4 n)$ Hańćkowiak, Karoński, Panconesi [SODA'98, PODC'99] **Our Result**

deterministic $O(log^2 \Delta \cdot log n)$ -round Maximal Matching

improving over

 $O(\log^4 n)$ Hańćkowiak, Karoński, Panconesi [SODA'98, PODC'99]

 $O(\Delta + \log^* n)$ Panconesi, Rizzi [DIST'01]

Overview of Results

Maximal Matching

- Maximal Matching
- Randomized Maximal Matching

Approximate Matching

- $(2 + \varepsilon)$ Approximate Maximum Matching
- $(2 + \varepsilon)$ Approximate Maximum Weighted Matching
- $(2 + \varepsilon)$ Approximate Maximum B-Matching
- $(2 + \varepsilon)$ Approximate Maximum Weighted B-Matching
- $ε$ Maximal Matching
- $(2 + \varepsilon)$ Approximate Minimum Edge Dominating Set

 $O(\log^2 \Delta \cdot \log n)$ $O(\log^3 \log n + \log \Delta)$

$$
O\left(\log^2 \Delta \cdot \log \frac{1}{\epsilon} + \log^* n\right)
$$

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I) 4 - Approximate Fractional Matching

 $O(\log \Delta)$ rounds

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II) Rounding Fractional Bipartite Matching

 $O(\log^2\Delta)$ rounds, $O(1)$ loss

Fractional Maximum Matching	
\n $\max \sum_{e \in E} x_e$ \n	\n $\text{value of } v$ \n
\n $\text{s.t.} \sum_{e \in E(v)} x_e \leq 1$ \n	\n $\text{for all } v \in V$ \n
\n $x_e \in [0,1]$ \n	\n $\text{for all } e \in E$ \n

LOCAL Greedy Algorithm $x_e = 2^{-\lceil \log \Delta \rceil}$ for all $e \in E$ repeat until all edges are blocked mark half-tight nodes block its edges

double value of unblocked edges

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 $\overline{2}$

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II) Rounding Fractional Bipartite Matching $O(\log^2 \Delta)$ **rounds,** $O(1)$ **loss**

using Locally Balanced Splitting, inspired by *Hańćkowiak, Karoński, Panconesi* [SODA'98,PODC'99]

Iterated Factor-2-Rounding using Locally Balanced Splitting

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Locally Balanced Splitting:

2-edge-coloring so that every node roughly balanced

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1

2

Iterated Factor-2-Rounding using Locally Balanced Splitting Locally Balanced Splitting: 2-edge-coloring so that every node roughly balanced 1 16 8 1 1 4

no constraint violated & no loss in total value (i.e., **perfect rouding**)

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Perfect Splitting not possible in case of…

Repeat until all edges colored pick arbitrary cycle alternate \square \square

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LOCAL Almost-Perfect Splitting*

Decompose into edge-disjoint cycles In parallel, for all cycles

- A) **Short cycles** of length $O(\log \Delta)$
	- alternate \blacksquare
- B) **Long cycles**
	- chop at length $\Theta(\log \Delta)$
	- set boundary to 0

alternate \Box in between

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* by *Hańćkowiak, Karoński, Panconesi* [SODA'98,PODC'99] in $O(\log \Delta)$

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Over all O() **rounding iterations, total loss still constant! * bipartite and even degree!**

loss

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Constant - Approximate Matching log 2 Δ rounds $O(\log^2 \Delta)$ rounds

Maximal

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\boldsymbol{O}\bigl(\log^2\boldsymbol{\varDelta}\cdot\log n\bigr)
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maximum matching size in remainder graph decreases by constant factor

Constant - Approximate Matching $O(\log^2 \Delta)$ **rounds**

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maximum matching size in remainder graph decreases by constant factor

after $O(log n)$ **iterations, maximum matching size is 0, hence graph empty**

Open Question: $O(log Δ \cdot log n)$?

What is Locality of Maximal Matching?

Thank you!

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