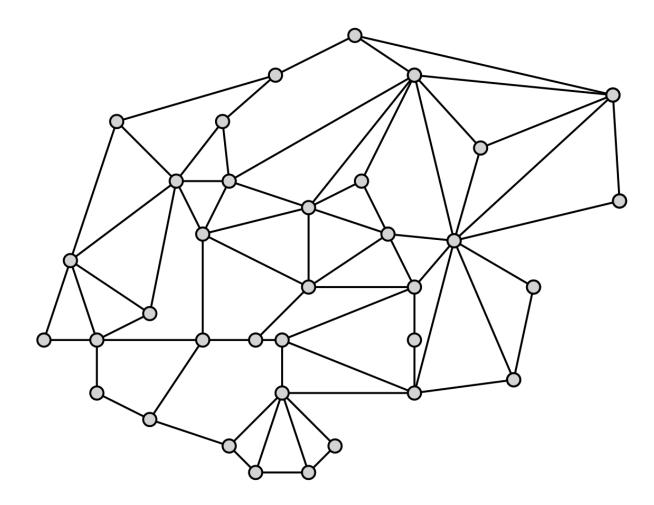
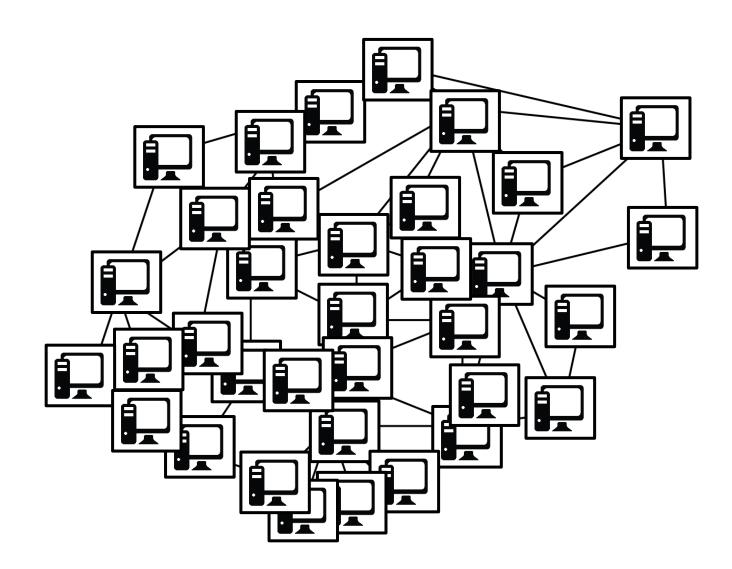
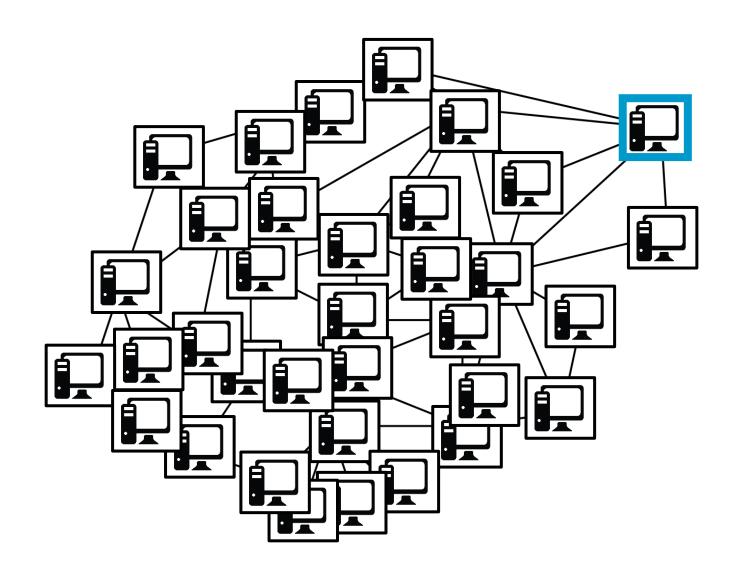


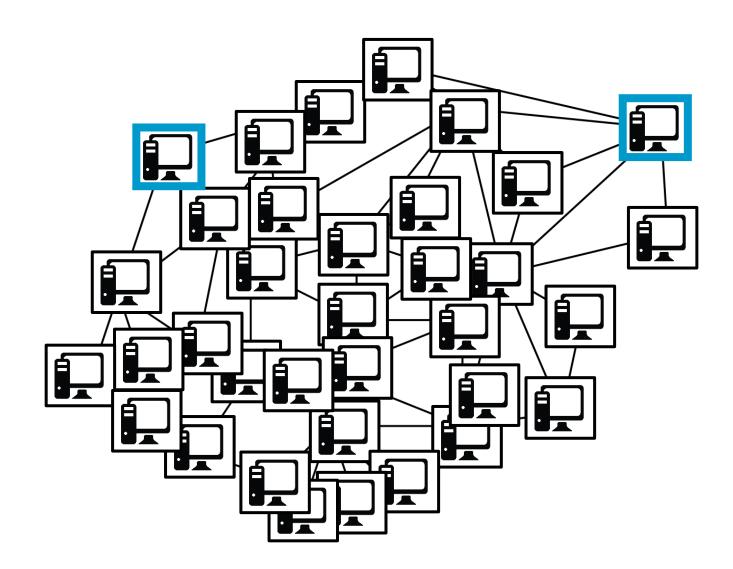
# The Locality of Maximal Matching

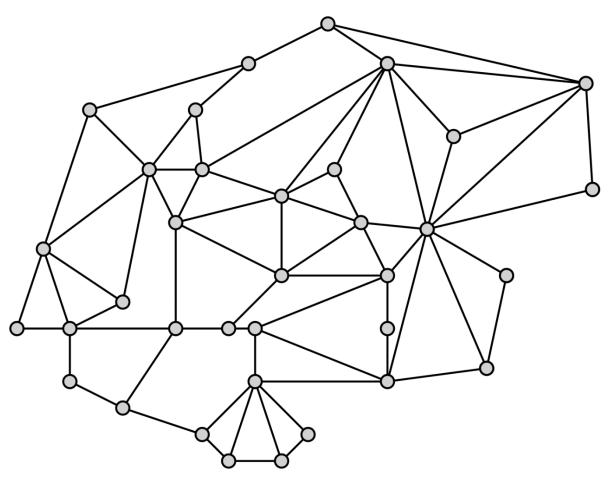
Manuela Fischer ETH Zurich





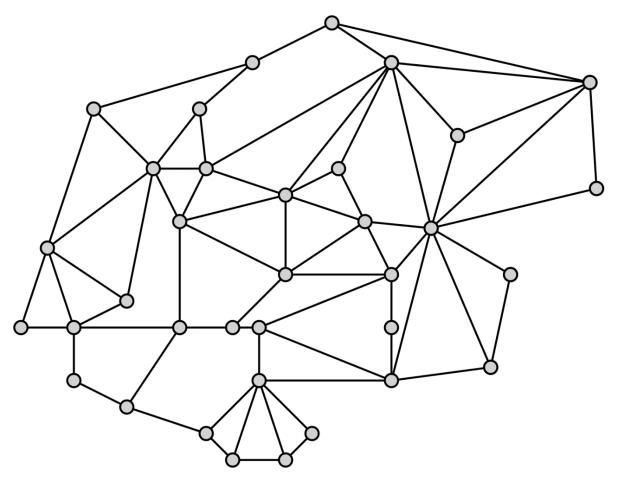






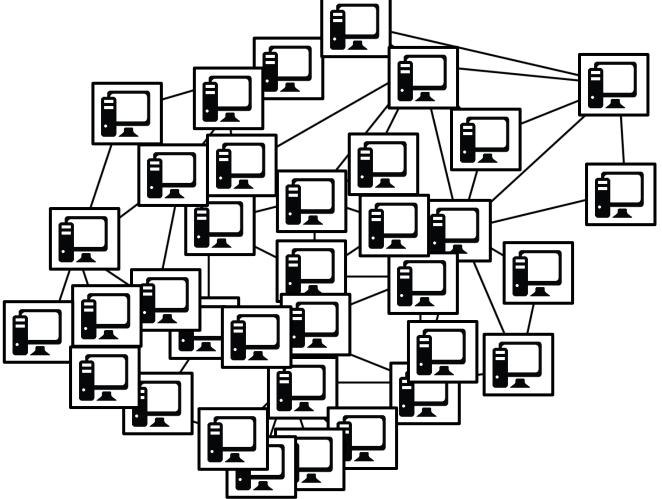
#### standard synchronous message-passing model of distributed computing

• undirected graph G = (V, E), n nodes, maximum degree  $\Delta$ 



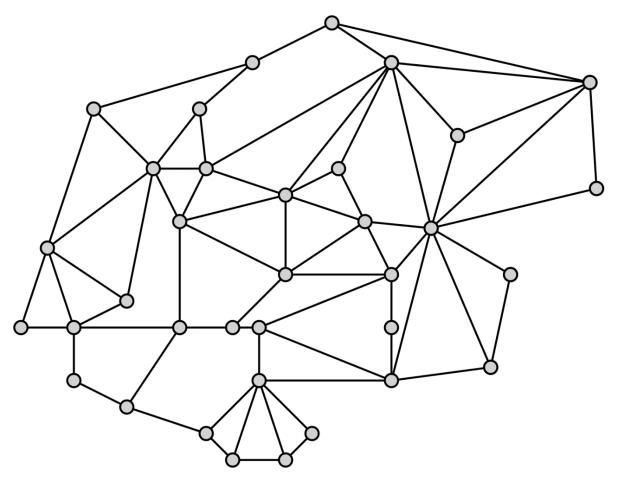
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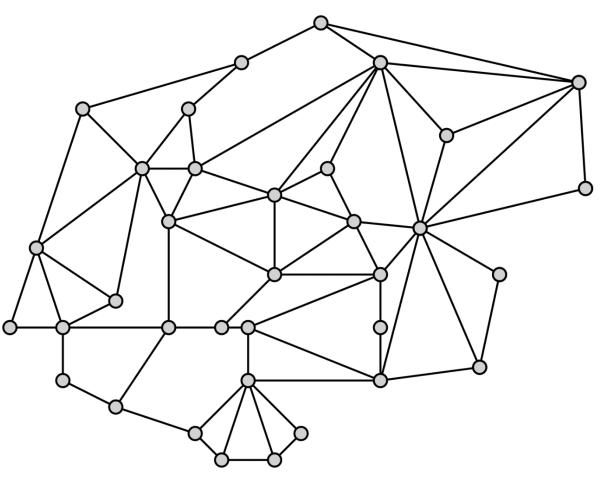


#### standard synchronous message-passing model of distributed computing

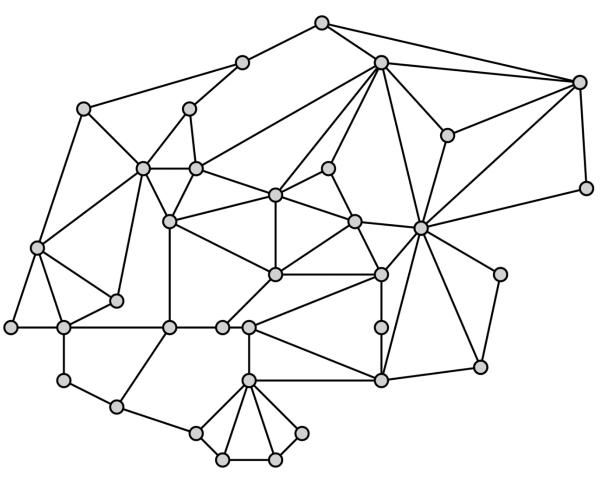
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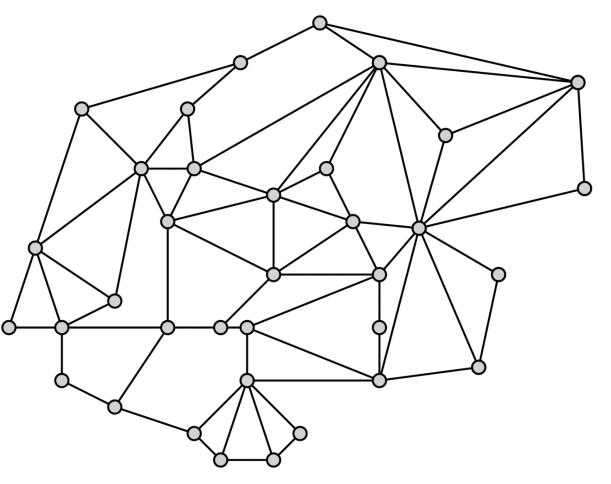
- undirected graph G = (V, E), n nodes, maximum degree  $\Delta$
- each round, every node
  - receives messages (sent in previous round)
  - performs some computation
  - sends message to all its neighbors



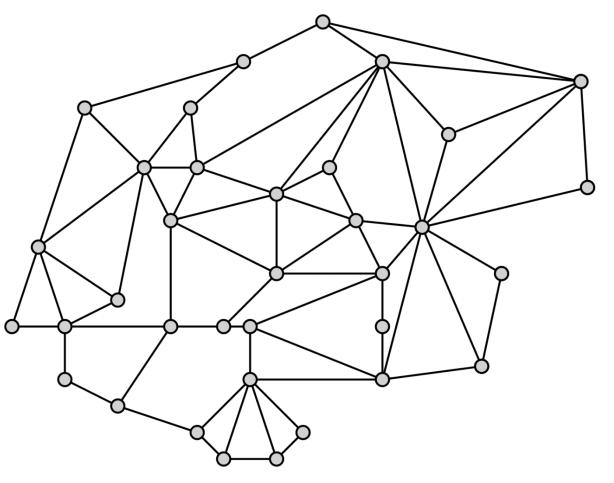
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- unbounded message size



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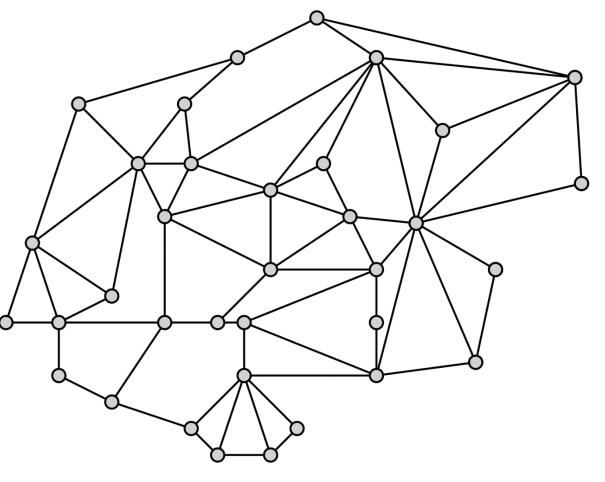


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- unbounded message size
- unbounded computation
- Round Complexity: number of rounds to solve the problem



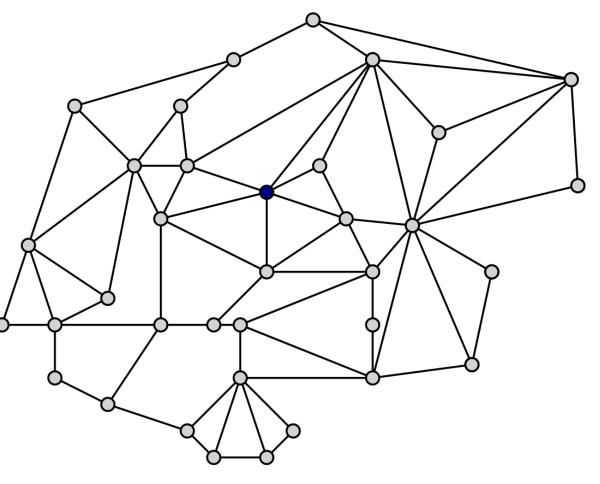
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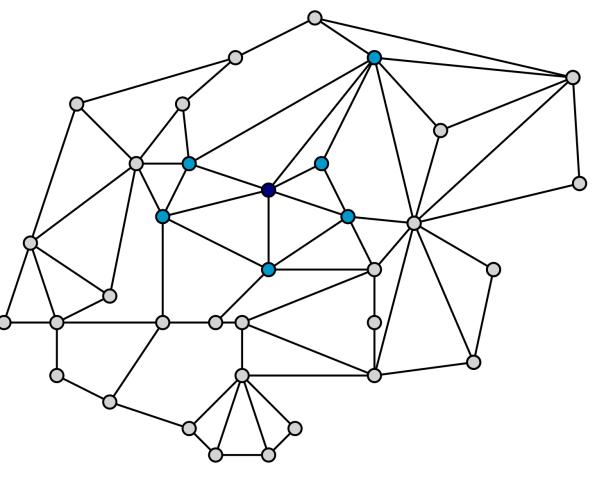
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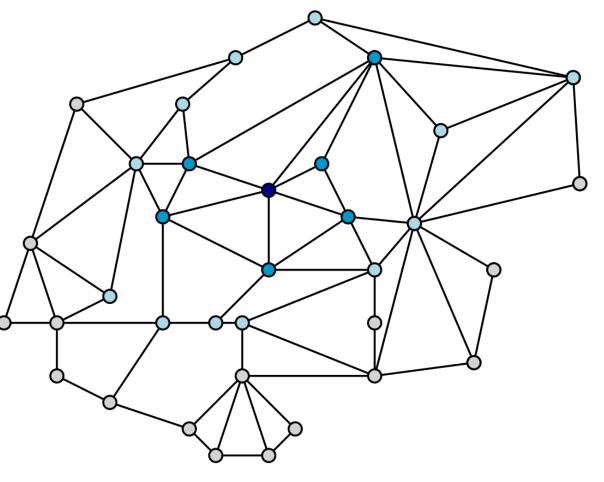
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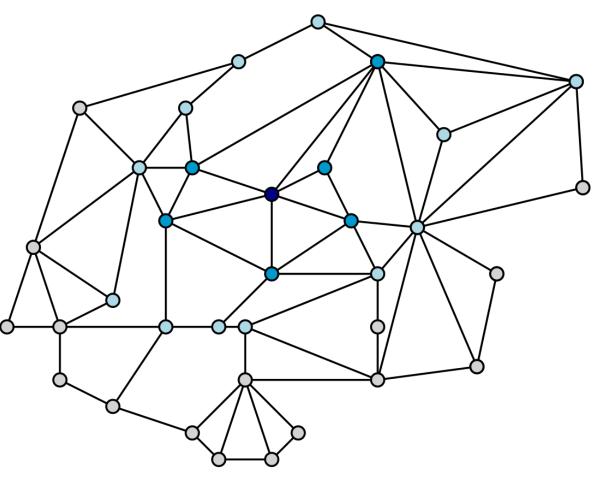
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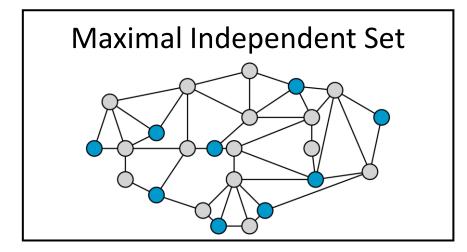


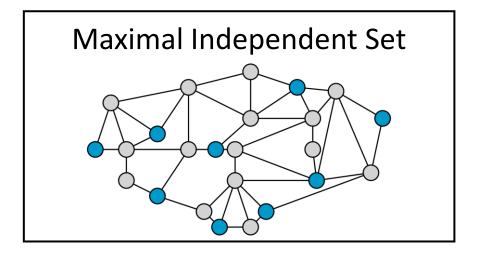
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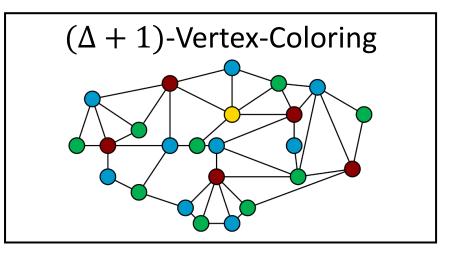
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  round complexity of a problem in the LOCAL

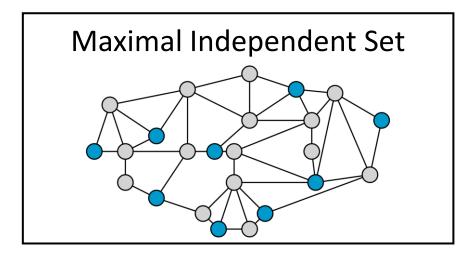
round complexity of a problem in the LOCAL model characterizes its locality every problem is trivially solvable in *O*(diameter) rounds

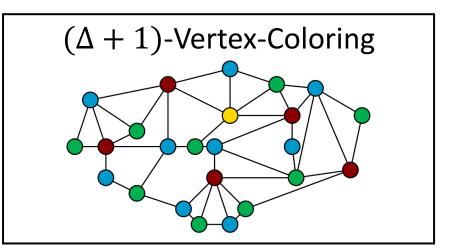


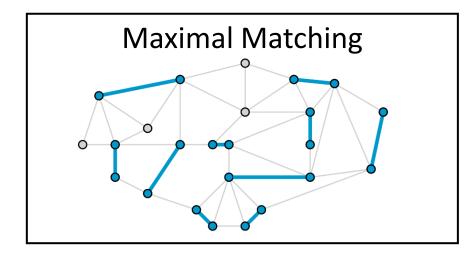


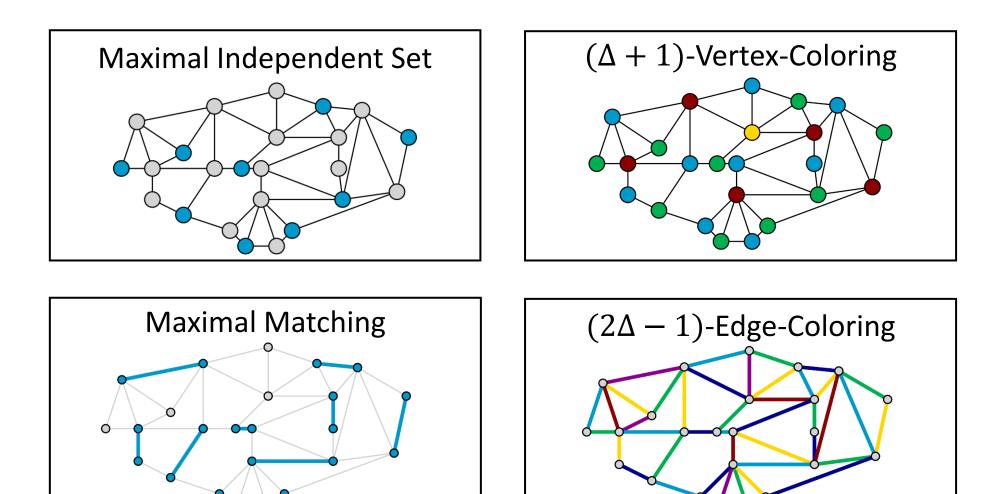


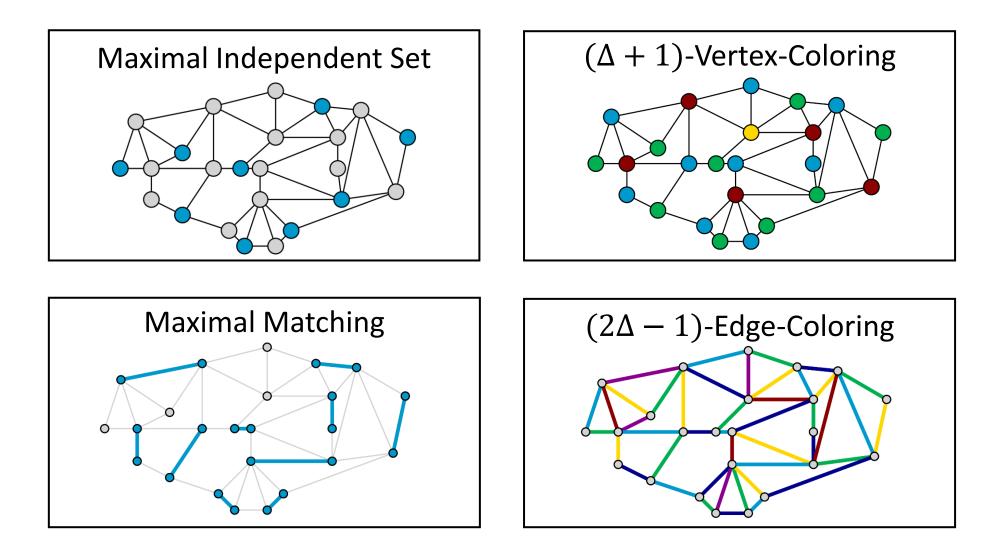




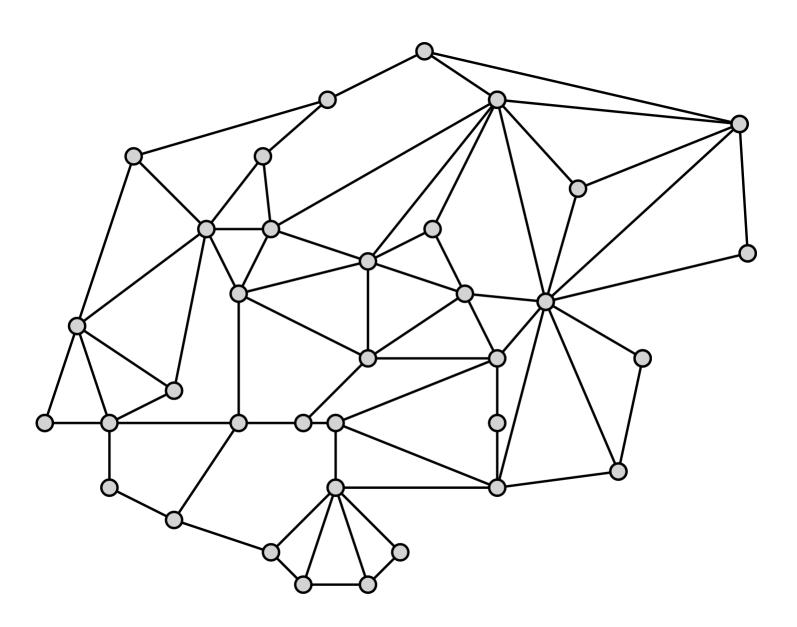


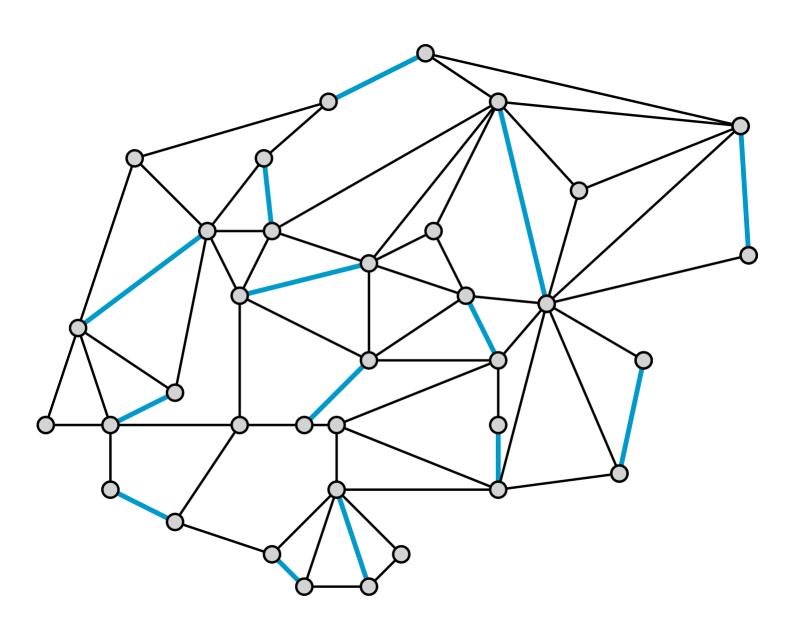




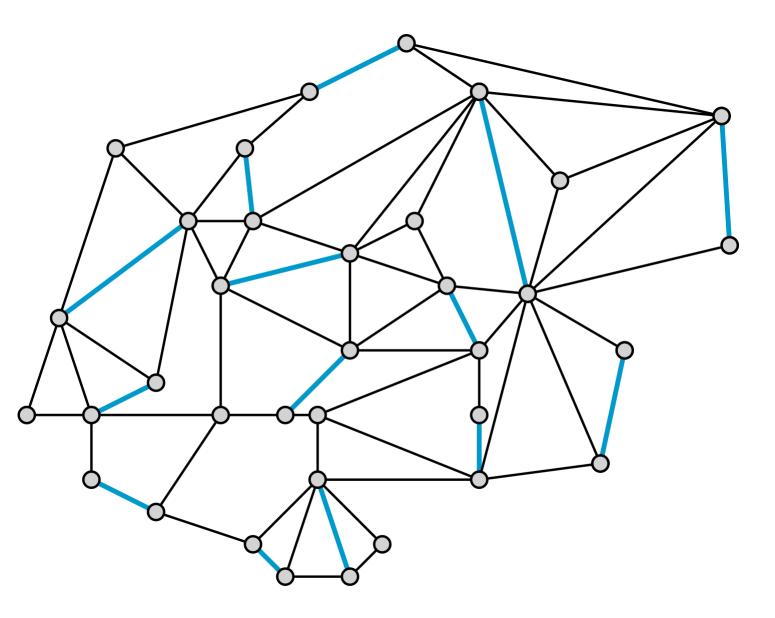


#### Easy centralized problems: greedy solutions.



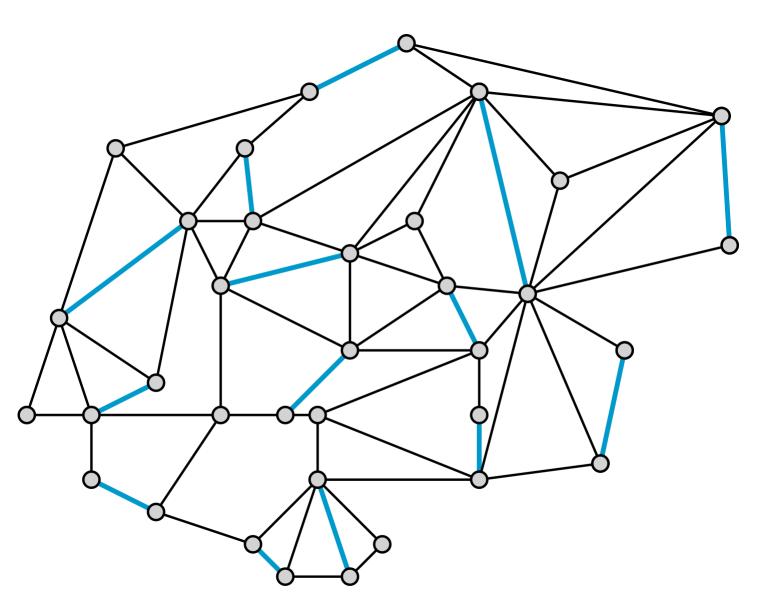


Matching: set of non-incident edges



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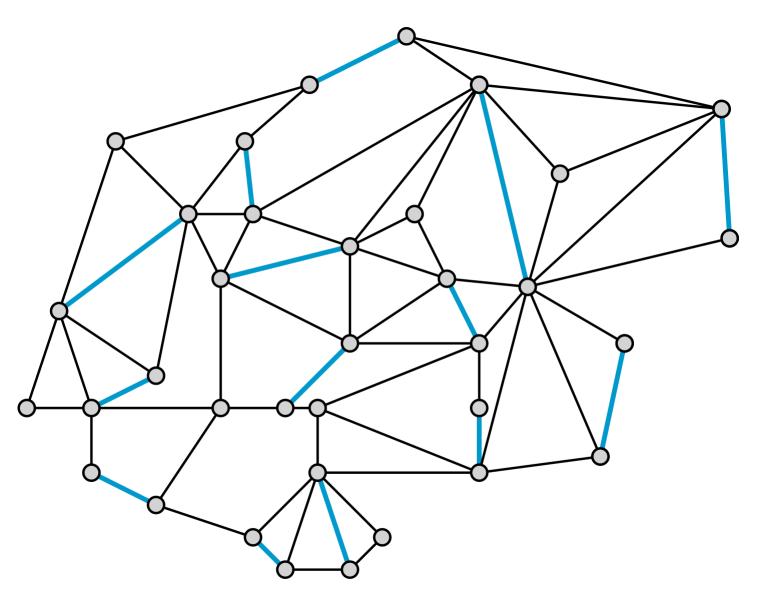
Maximal: no edge can be added



<u>Matching:</u> set of non-incident edges

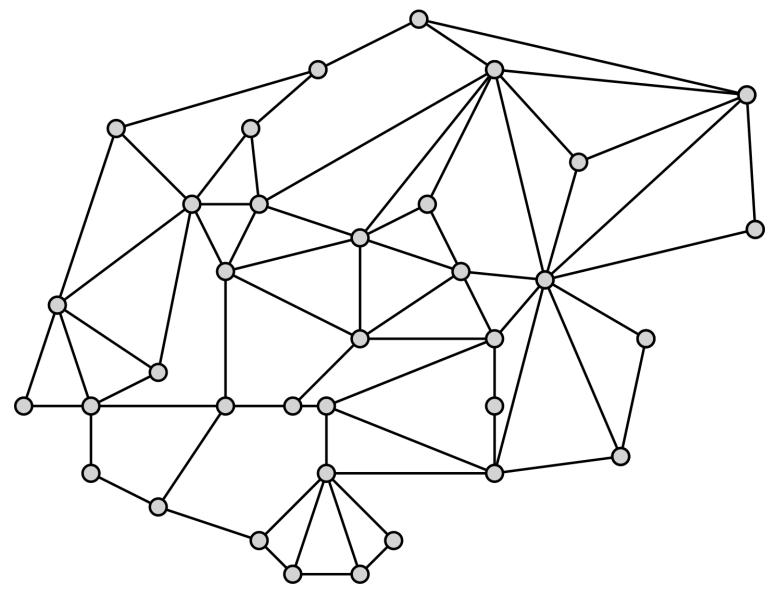
<u>Maximal:</u> no edge can be added

greedy property!

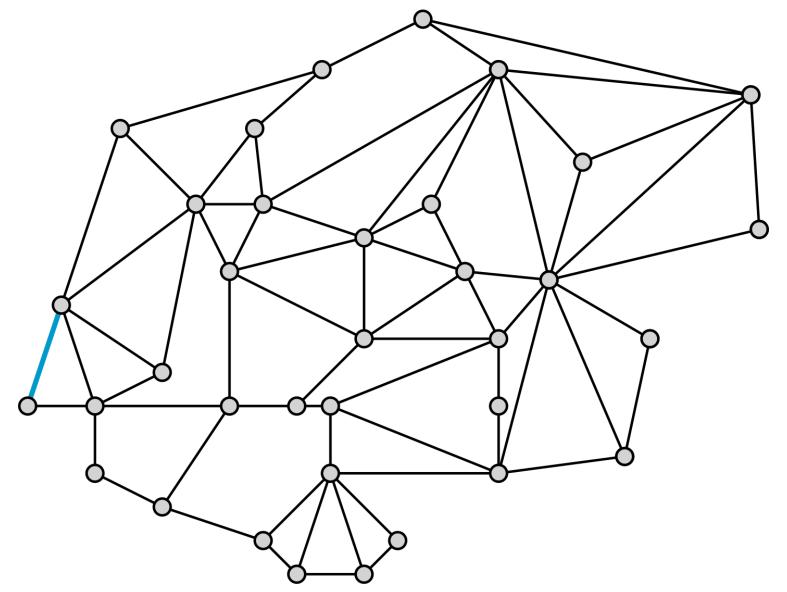


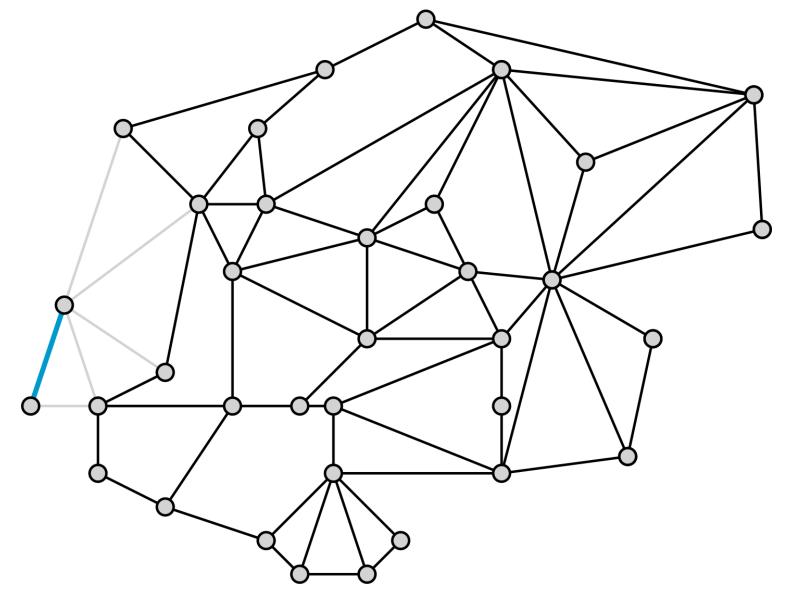
### **Centralized (Sequential) Algorithm**

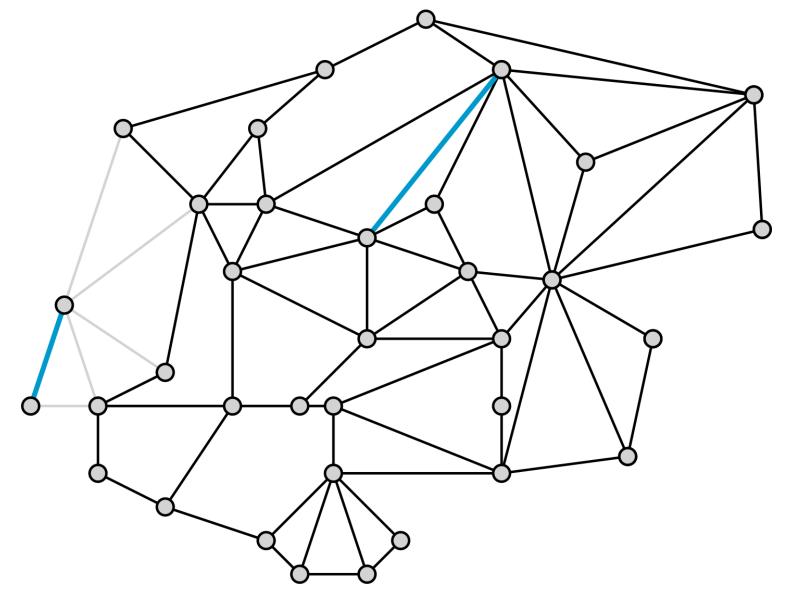
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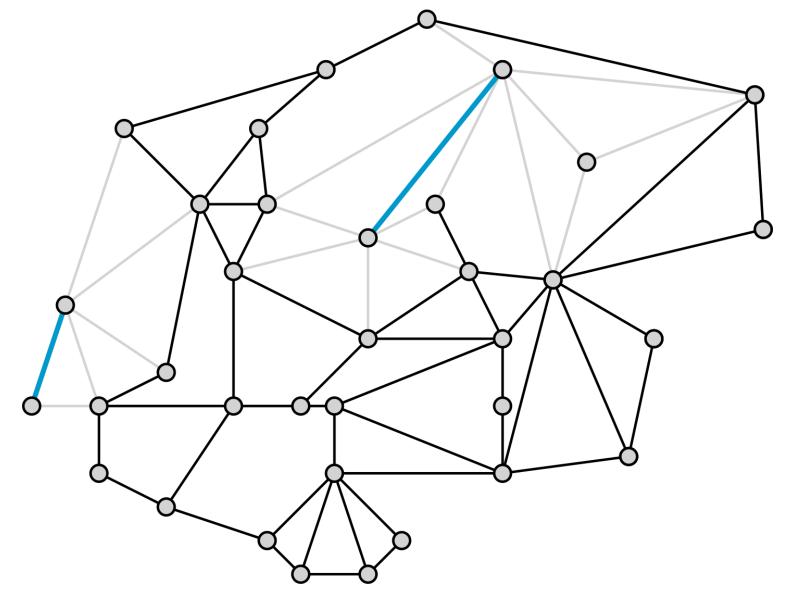


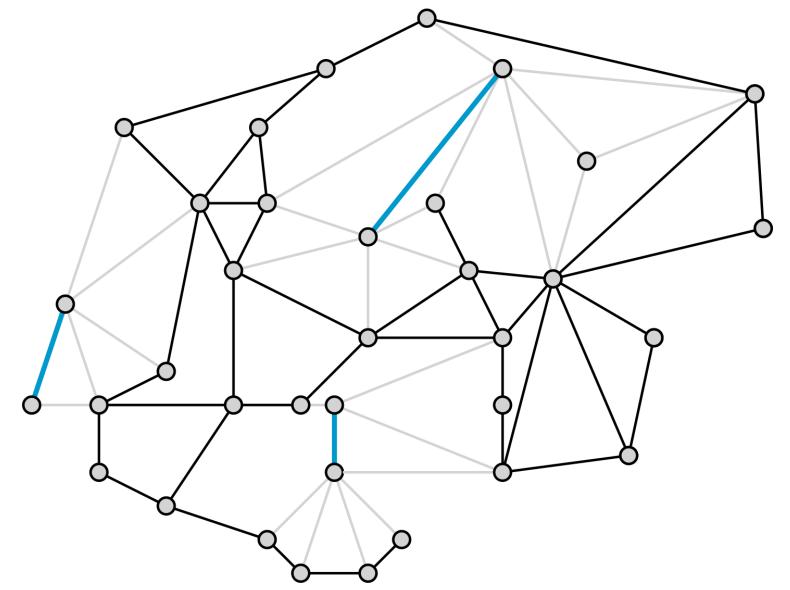
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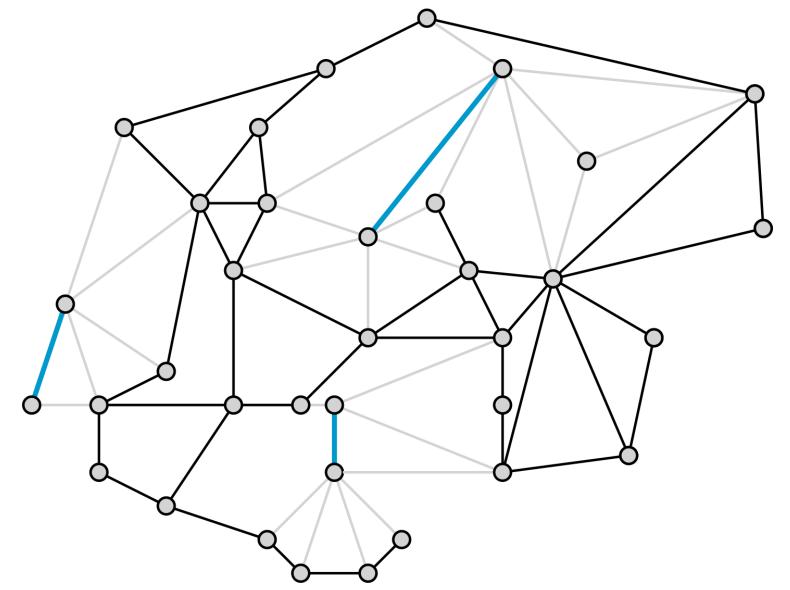


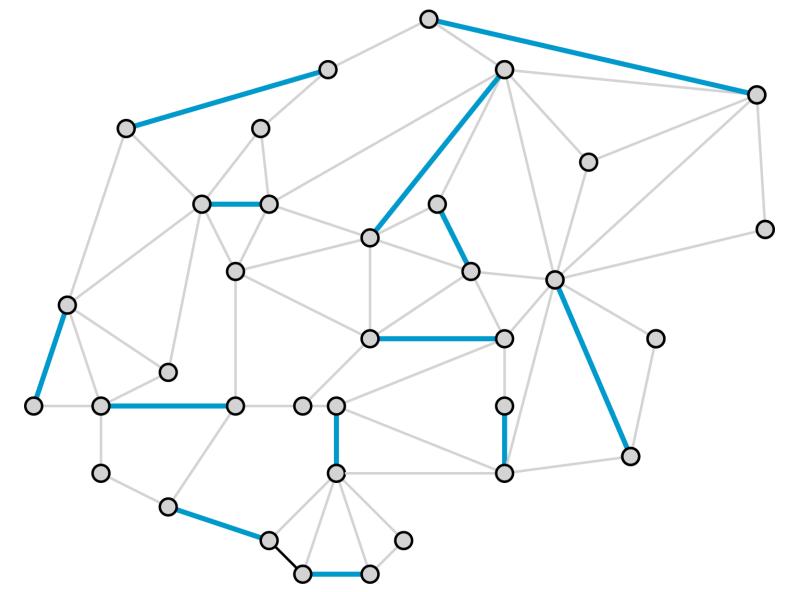


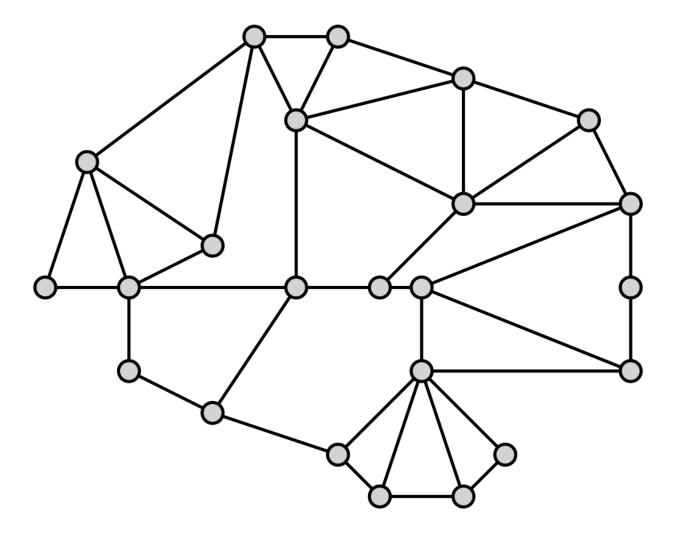


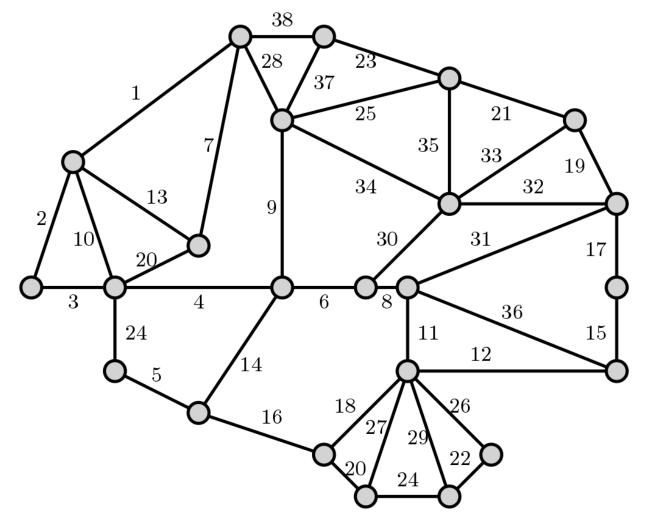


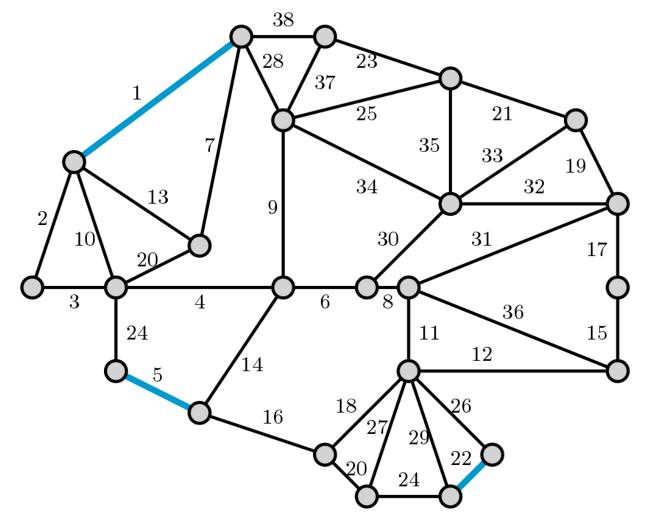


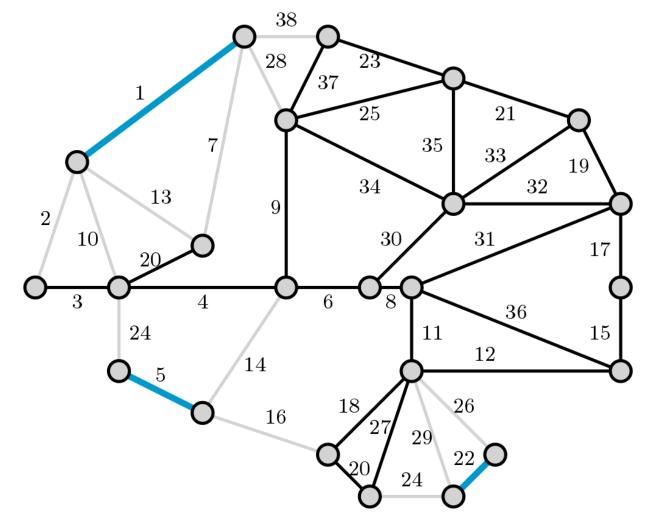


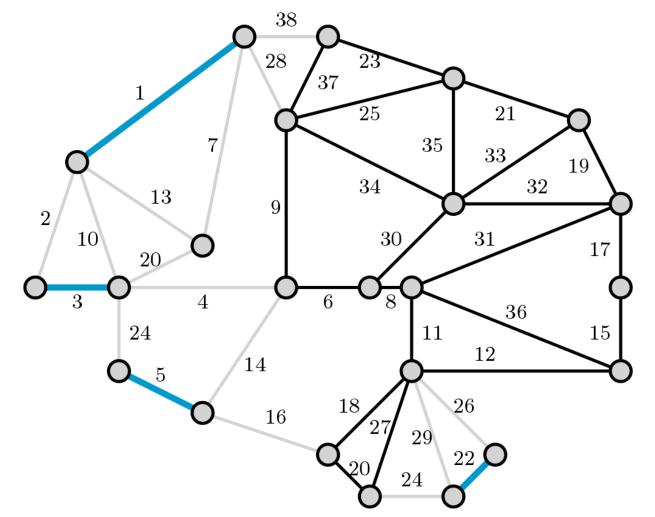


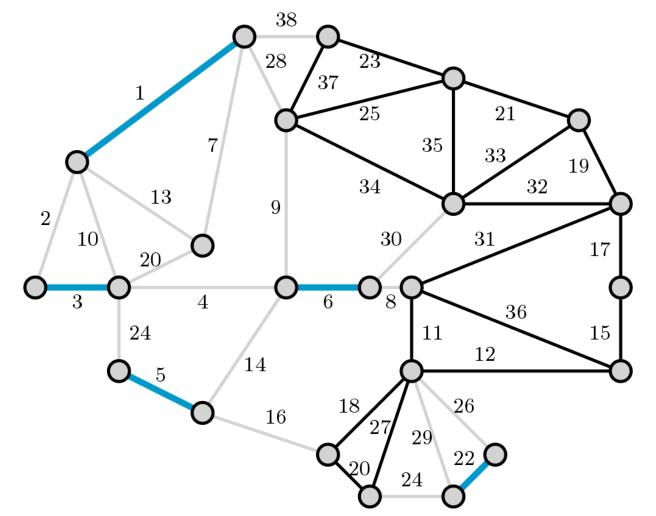


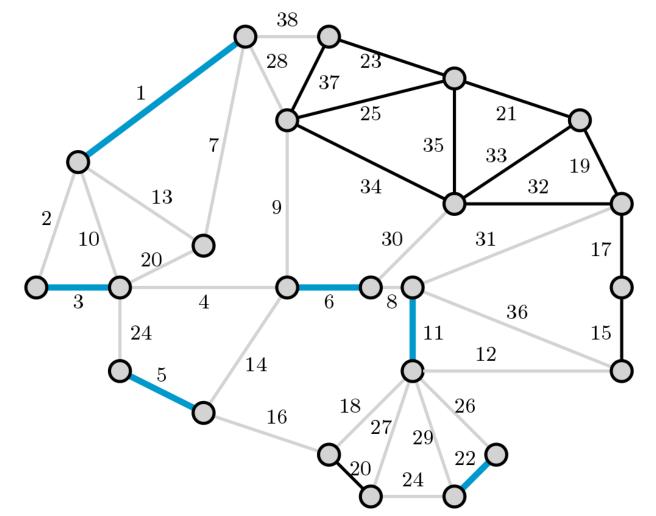


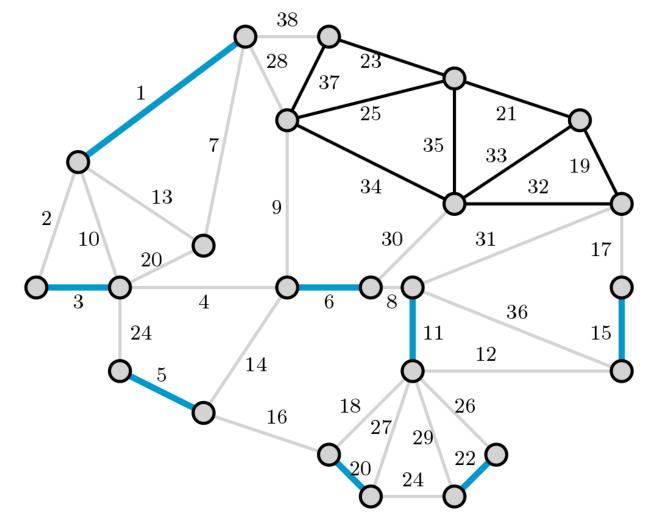


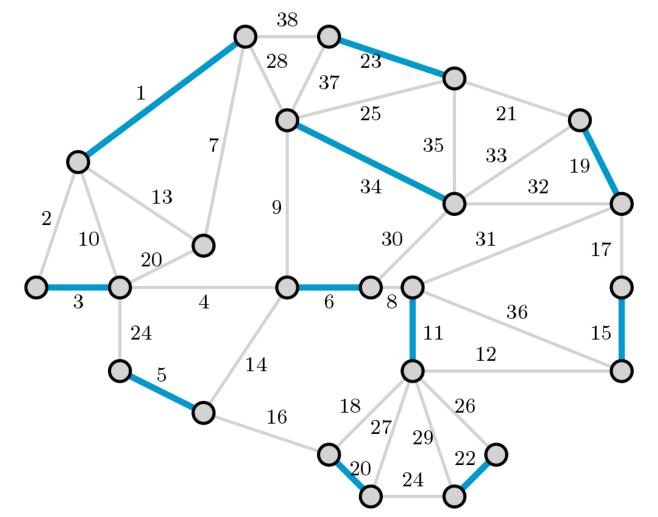


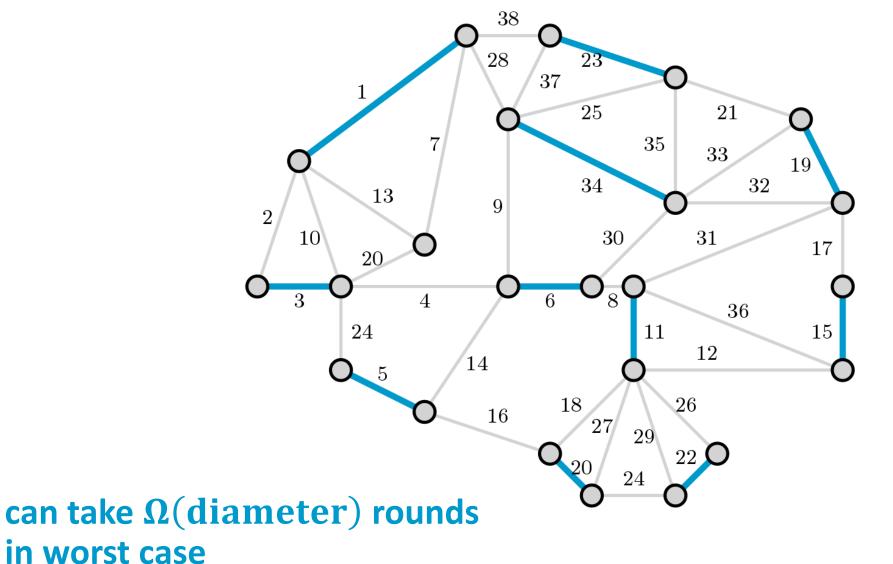


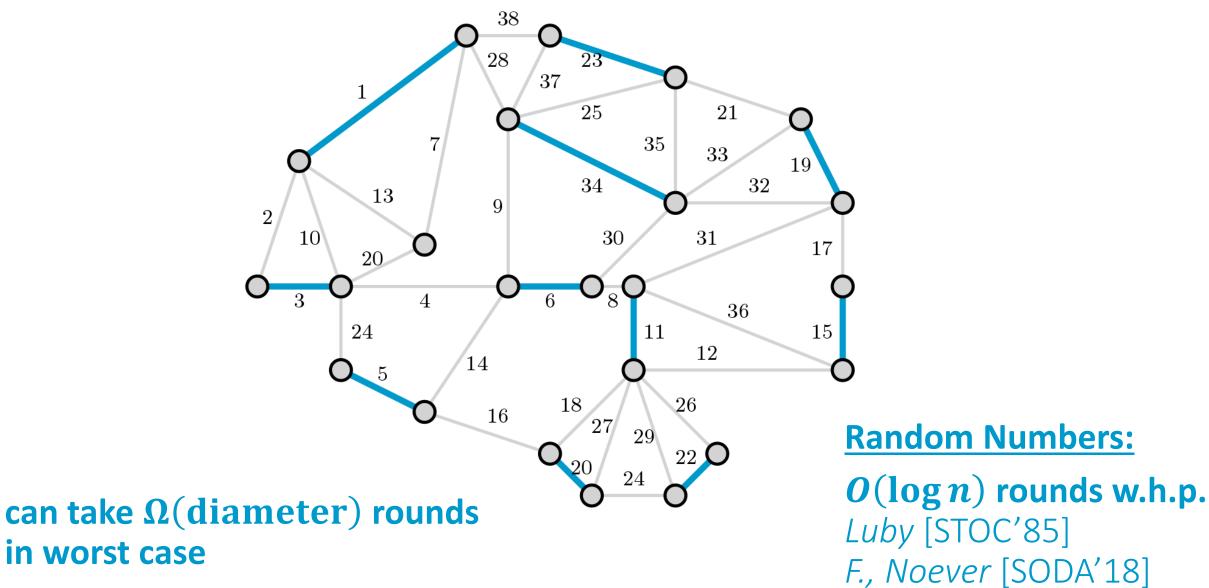


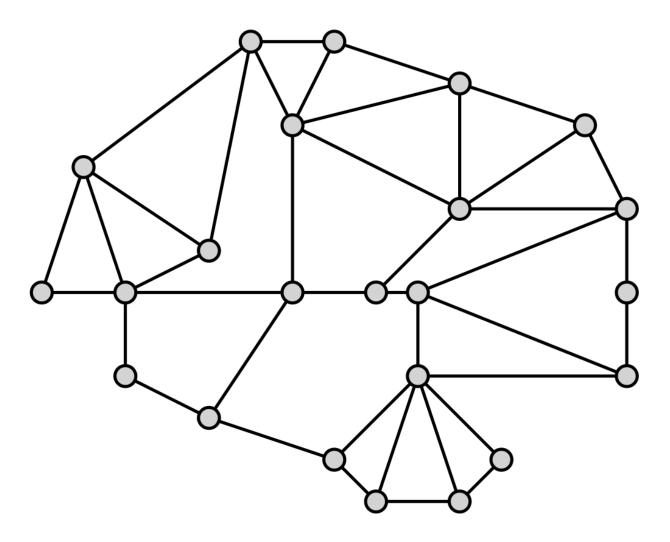


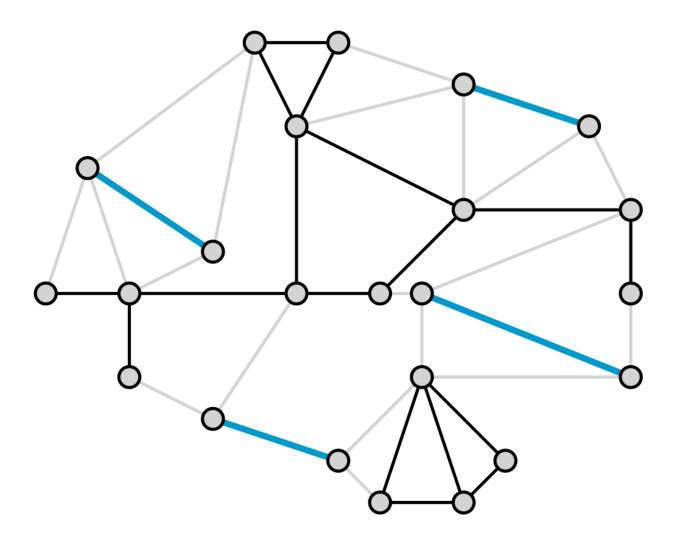


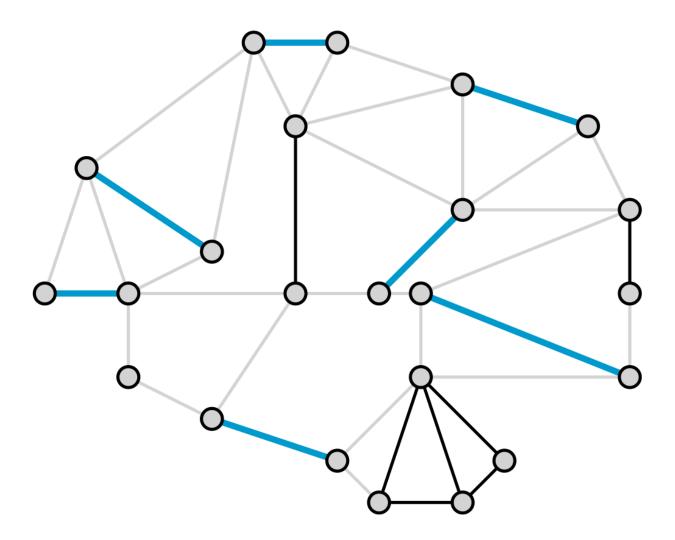


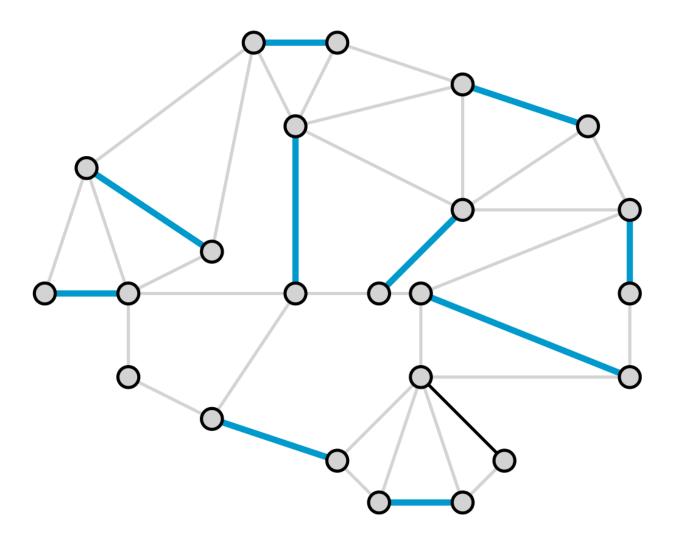


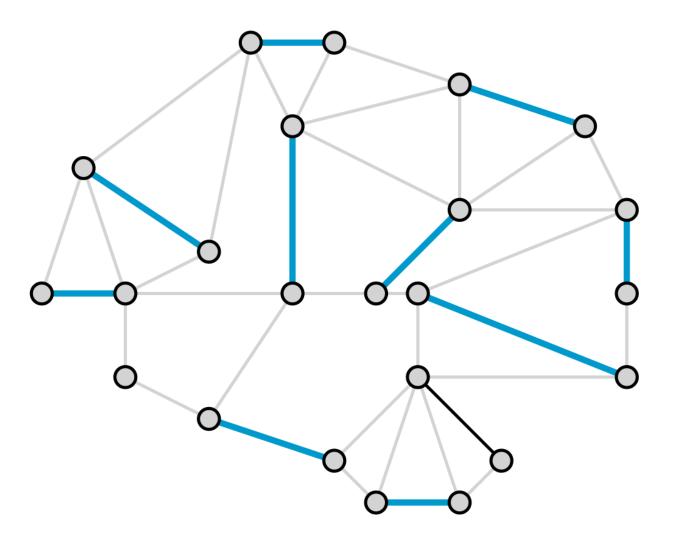




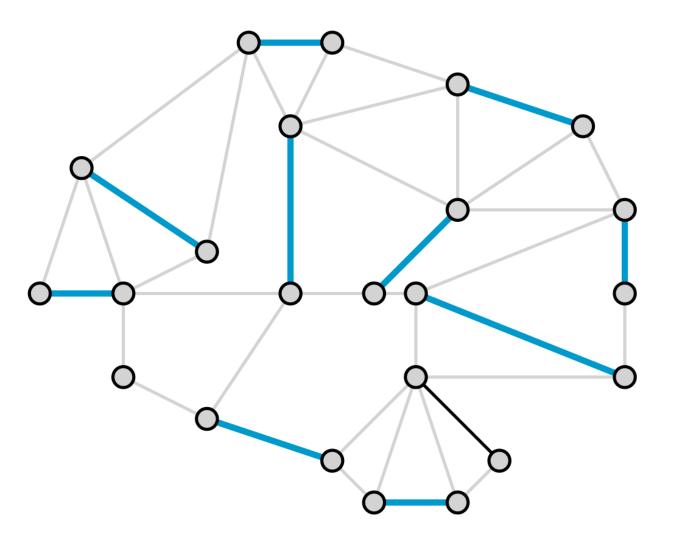








 $\mathbb{E}[$ #removed edges per round $] \geq c|E_i|$ 



 $\mathbb{E}[\text{#removed edges per round}] \ge c|E_i|$   $O(\log n)$  rounds w.h.p.

### **Our Result**



## <u>deterministic</u> $O(\log^2 \Delta \cdot \log n)$ -round Maximal Matching



## <u>deterministic</u> $O(\log^2 \Delta \cdot \log n)$ -round Maximal Matching

improving over

**Our Result** 

# <u>deterministic</u> $O(\log^2 \Delta \cdot \log n)$ -round Maximal Matching

improving over

*O*(log<sup>4</sup> *n*) Hańćkowiak, Karoński, Panconesi [SODA'98, PODC'99] **Our Result** 

# <u>deterministic</u> $O(\log^2 \Delta \cdot \log n)$ -round Maximal Matching

improving over

*O*(log<sup>4</sup> *n*) Hańćkowiak, Karoński, Panconesi [SODA'98, PODC'99]

 $O(\Delta + \log^* n)$ Panconesi, Rizzi [DIST'01]

#### **Overview of Results**

#### **Maximal Matching**

- Maximal Matching
- Randomized Maximal Matching

#### **Approximate Matching**

- $(2 + \varepsilon)$  Approximate Maximum Matching
- $(2 + \epsilon)$  Approximate Maximum Weighted Matching
- $(2 + \varepsilon)$  Approximate Maximum B-Matching
- $(2 + \varepsilon)$  Approximate Maximum Weighted B-Matching
- ε Maximal Matching
- $(2 + \varepsilon)$  Approximate Minimum Edge Dominating Set

 $O(\log^2 \Delta \cdot \log n)$  $O(\log^3 \log n + \log \Delta)$ 

$$O\left(\log^2 \Delta \cdot \log \frac{1}{\epsilon} + \log^* n\right)$$
$$O\left(\log^2 \Delta \cdot \log \frac{1}{\epsilon} + \log^* n\right)$$

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$$O\left(\log^2 \Delta \cdot \log \frac{1}{\varepsilon}\right)$$
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 $O(\log^2 \Delta)$  rounds

 $O(\log^2 \Delta)$  rounds

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#### I) 4 - Approximate Fractional Matching

 $O(\log \Delta)$  rounds

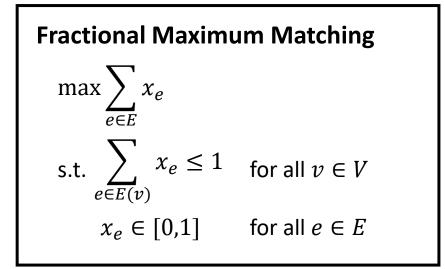
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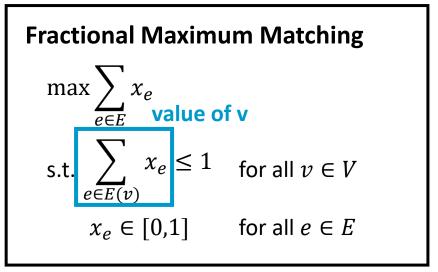
 $O(\log \Delta)$  rounds

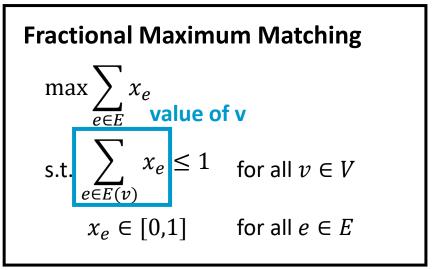
#### **II)** Rounding Fractional Bipartite Matching

 $O(\log^2 \Delta)$  rounds, O(1) loss

 $O(\log \Delta)$  rounds

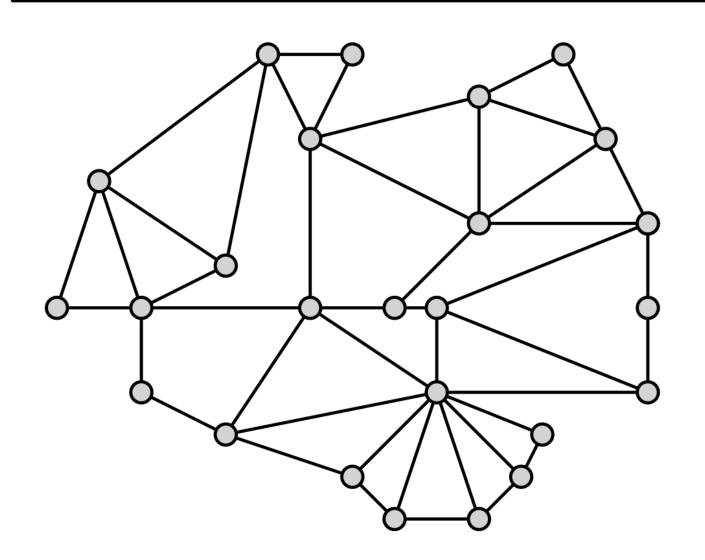






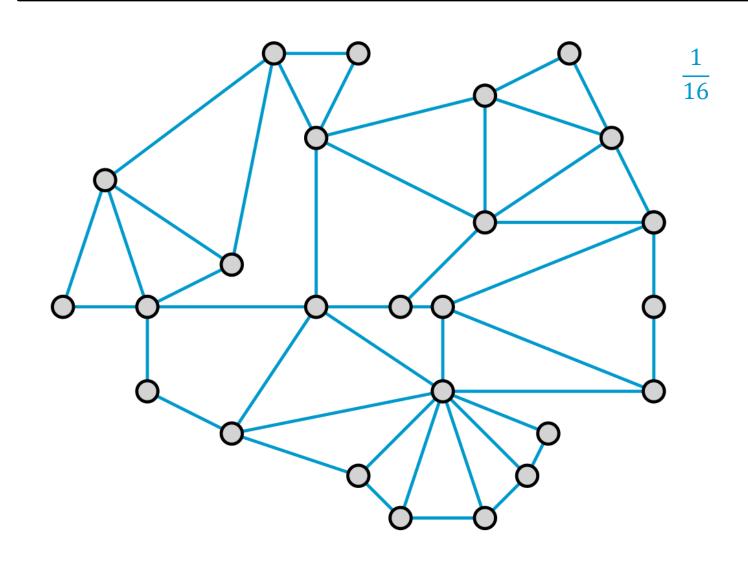
LOCAL Greedy Algorithm
$x_e = 2^{-\lceil \log \Delta \rceil}$ for all $e \in E$
repeat until all edges are blocked
mark half-tight nodes
block its edges
double value of unblocked edges

# $O(\log \Delta)$ rounds



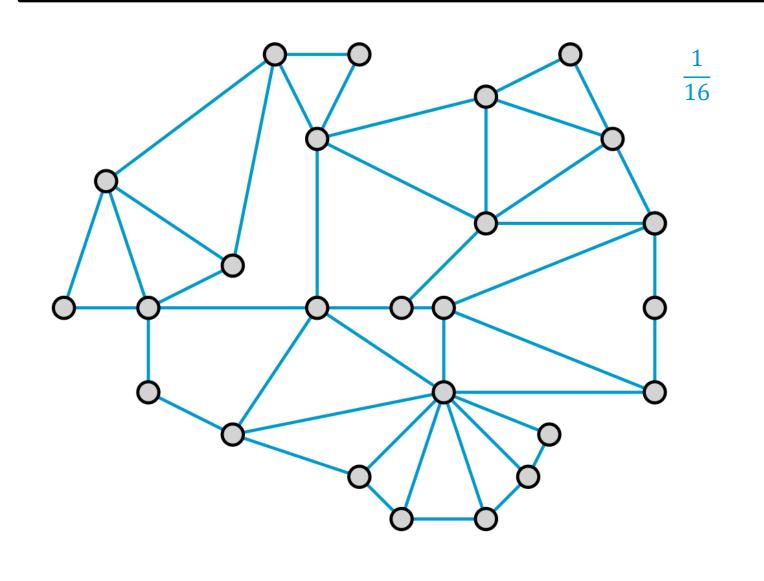
Fractional Maximum Matching
$$\max \sum_{e \in E} x_e$$
  
 $value of v$  $s.t.$  $\sum_{e \in E(v)} x_e \leq 1$   
 $x_e \in [0,1]$ for all  $v \in V$  $x_e \in [0,1]$ 

# LOCAL Greedy Algorithm $x_e = 2^{-\lceil \log \Delta \rceil}$ for all $e \in E$ repeat until all edges are blockedmark half-tight nodesblock its edgesdouble value of unblocked edges

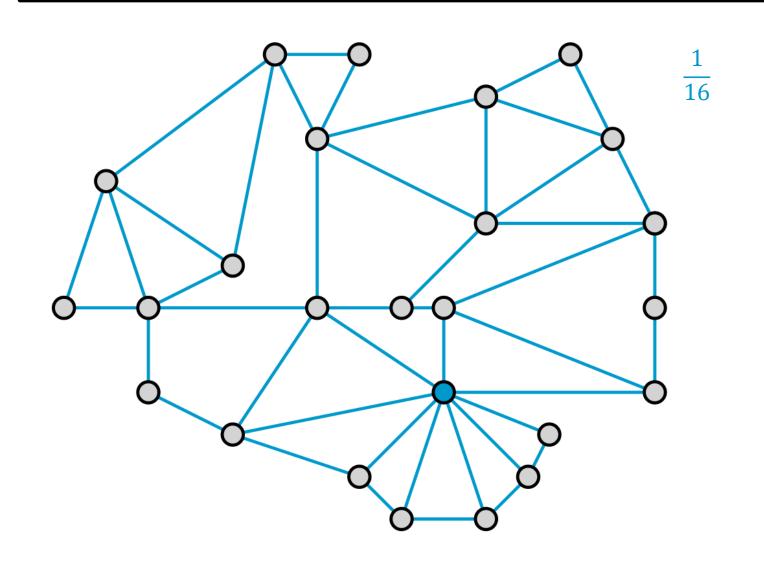


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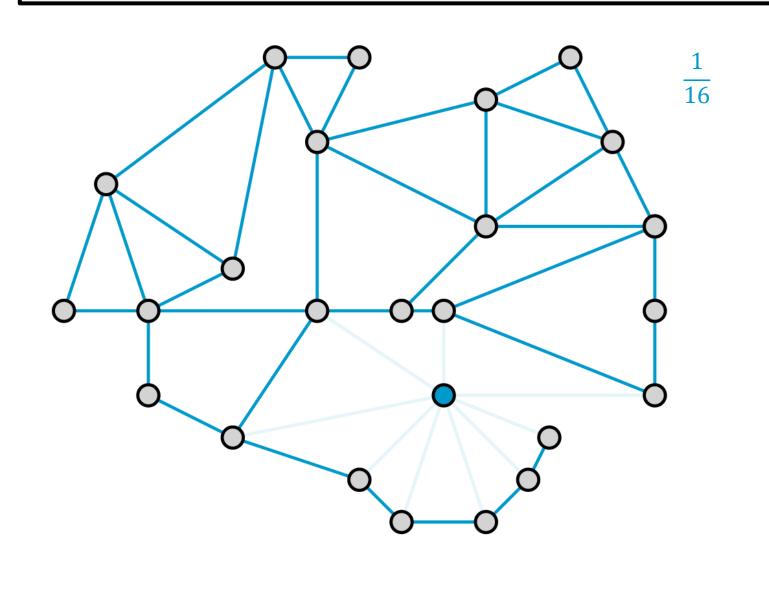
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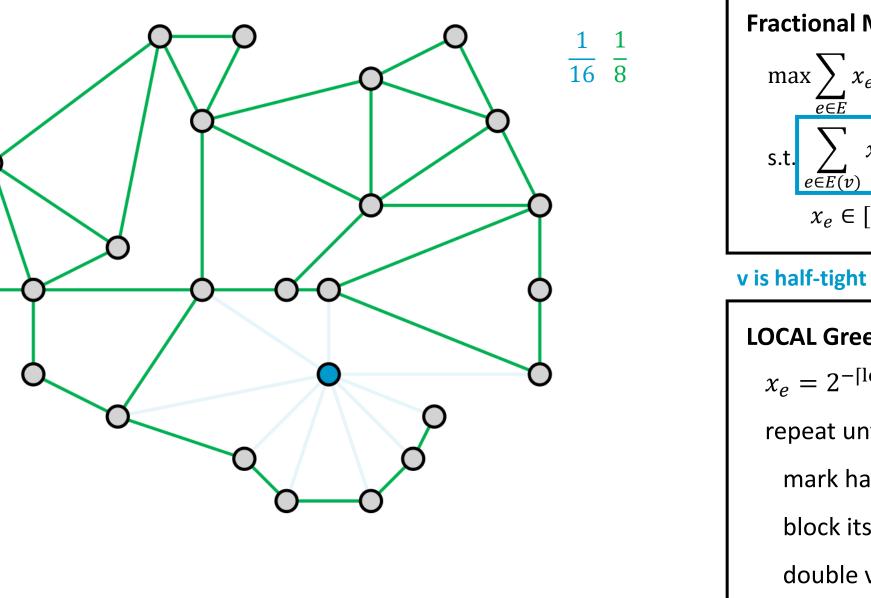
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<b>v</b> is half-tight if its value is $\geq \frac{1}{2}$
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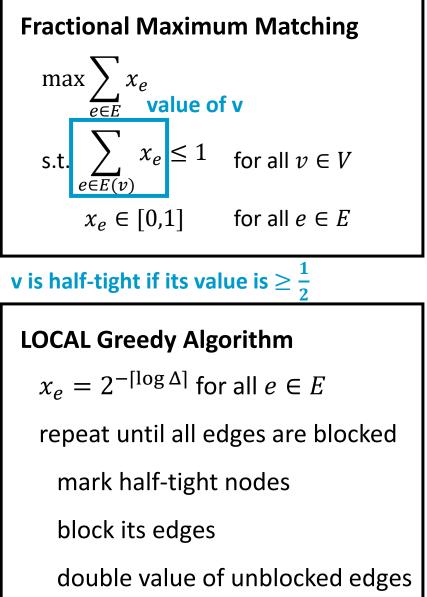


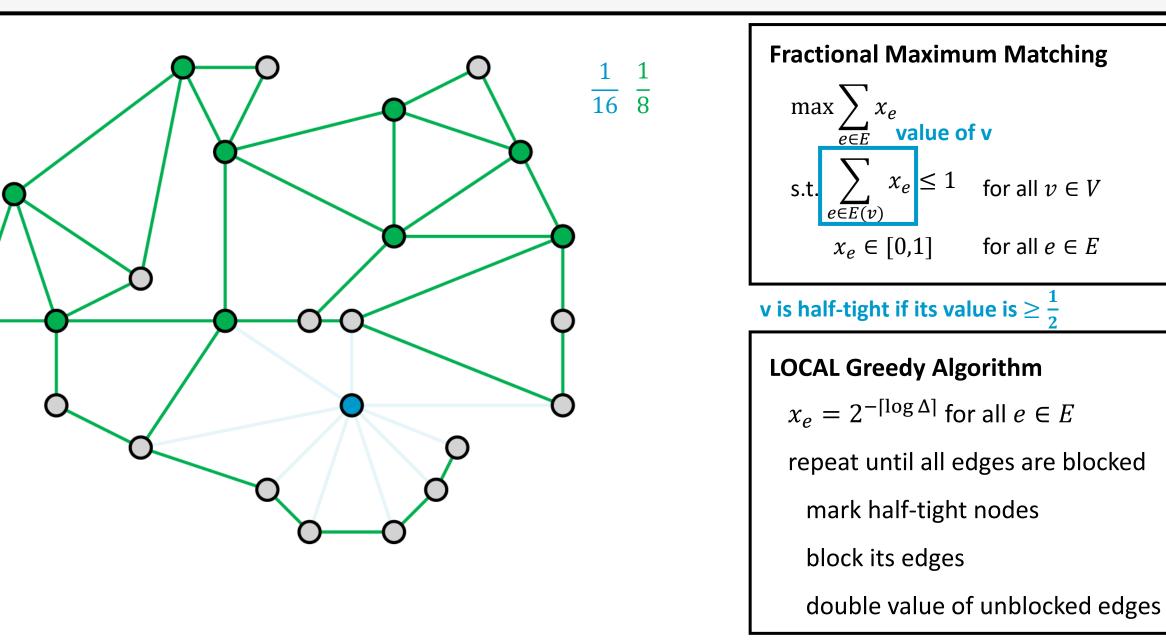
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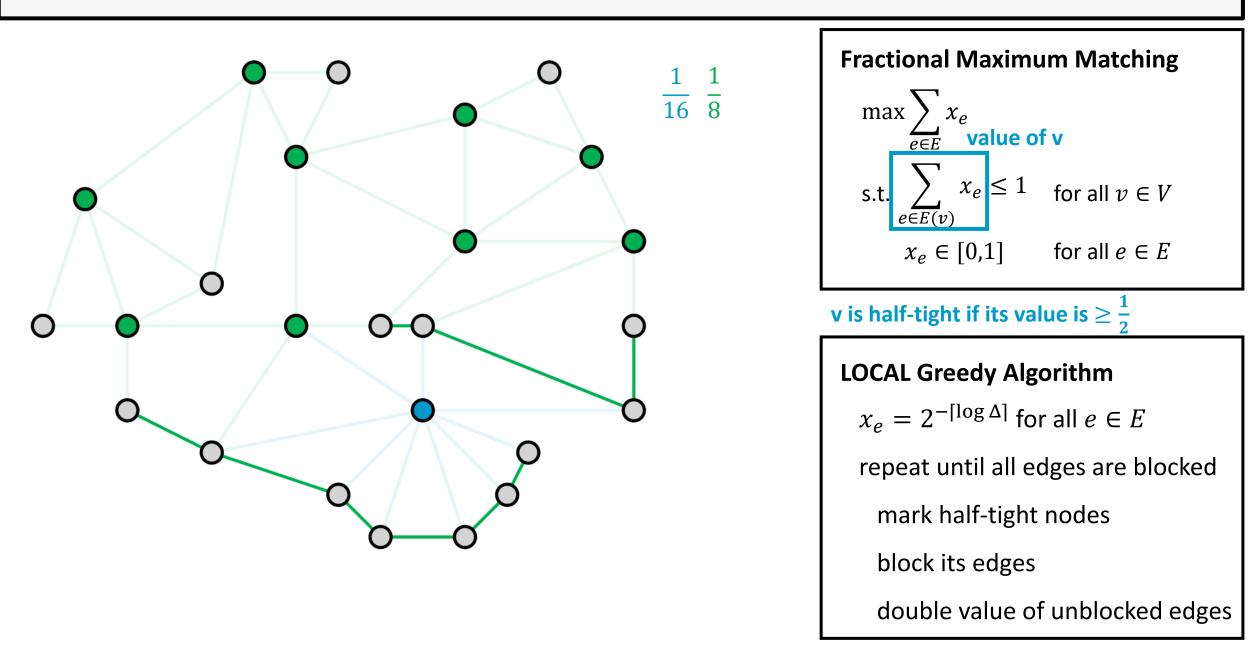


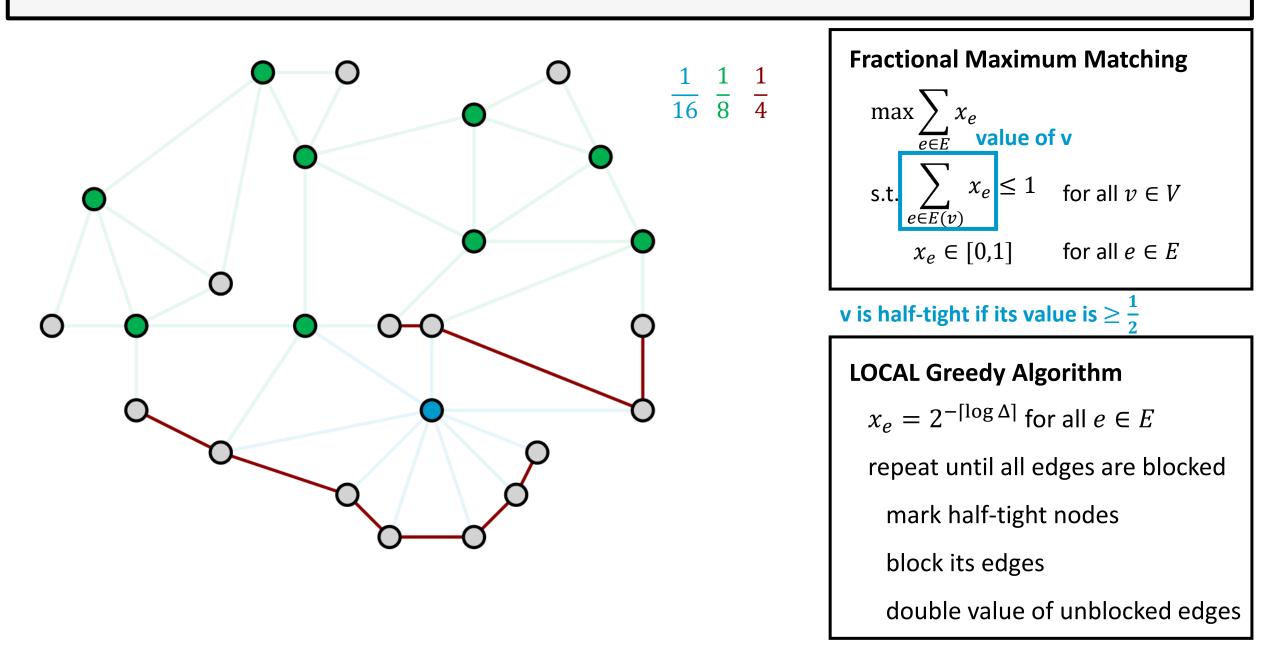
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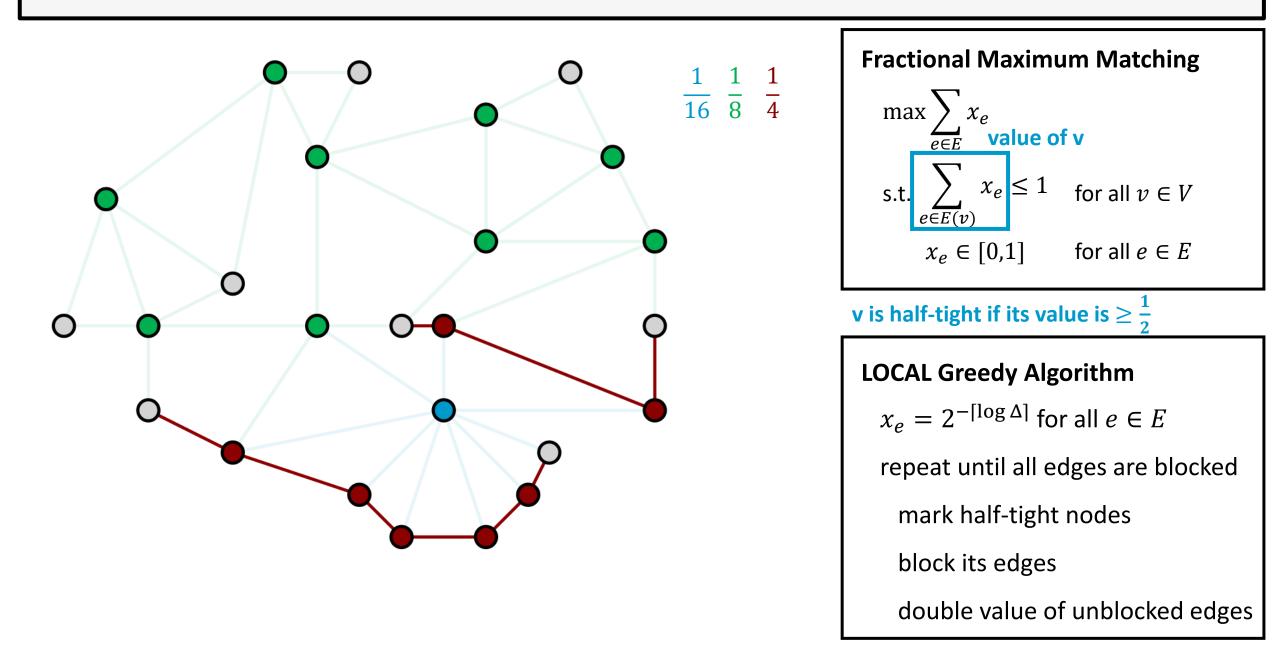


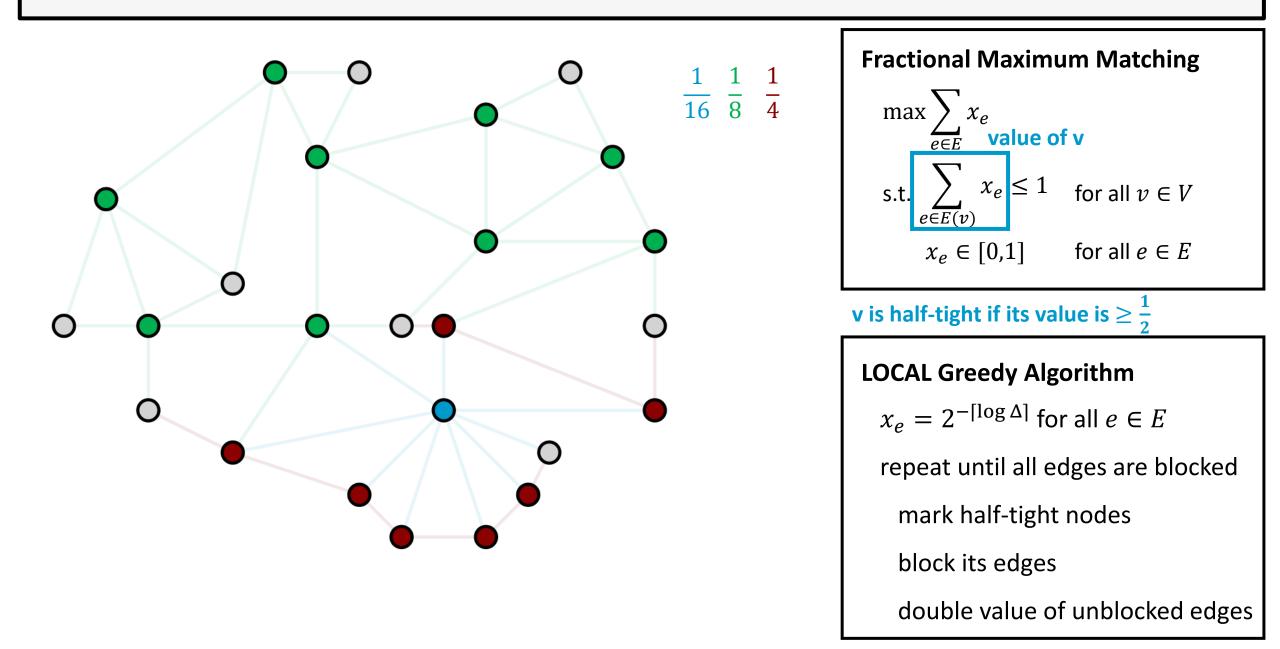










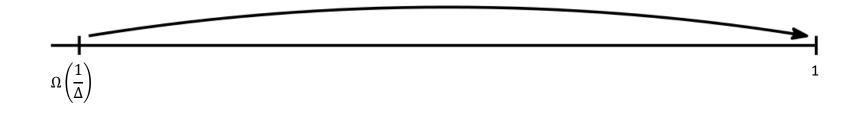


 $O(\log \Delta)$  rounds

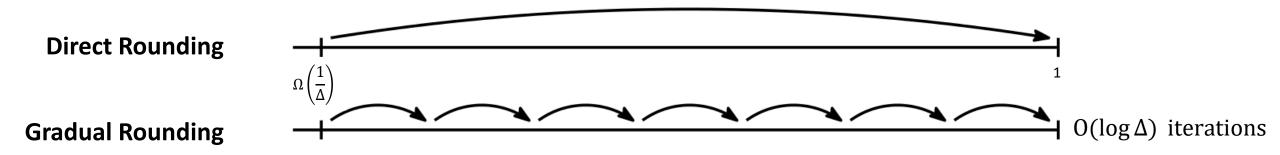
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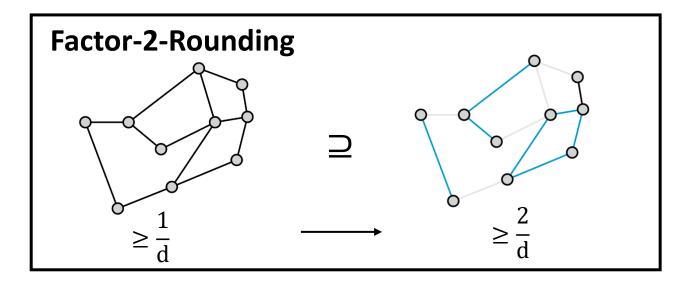


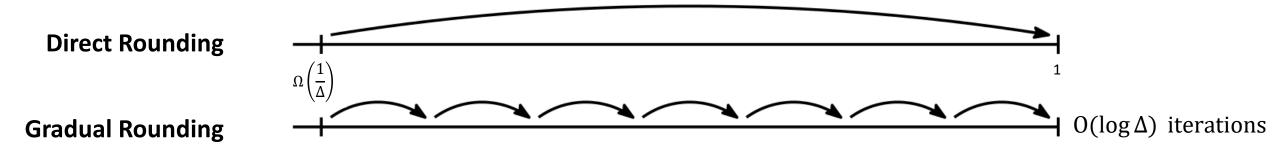


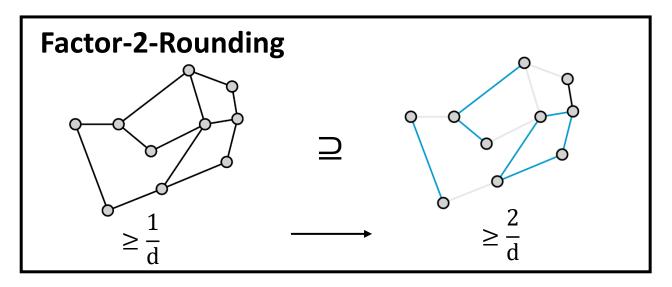


### **II)** Rounding Fractional Bipartite Matching

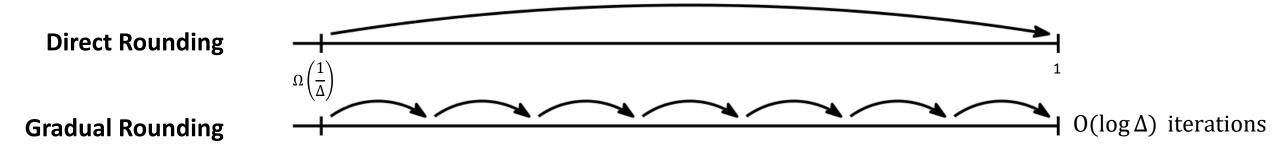
# $O(\log^2 \Delta)$ rounds, O(1) loss







using Locally Balanced Splitting, inspired by Hańćkowiak, Karoński, Panconesi [SODA'98,PODC'99]

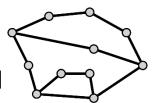


**Iterated Factor-2-Rounding using Locally Balanced Splitting** 

#### **Iterated Factor-2-Rounding using Locally Balanced Splitting**

#### **Locally Balanced Splitting:**

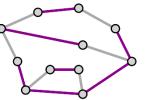
2-edge-coloring so that every node roughly balanced



#### Iterated Factor-2-Rounding using Locally Balanced Splitting

#### **Locally Balanced Splitting:**

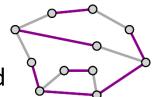
2-edge-coloring so that every node roughly balanced

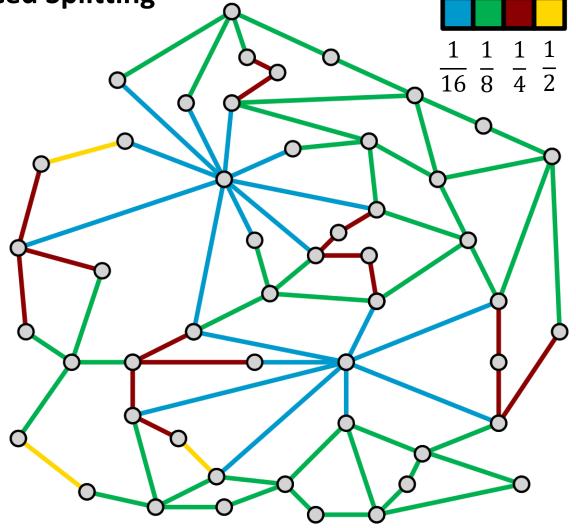


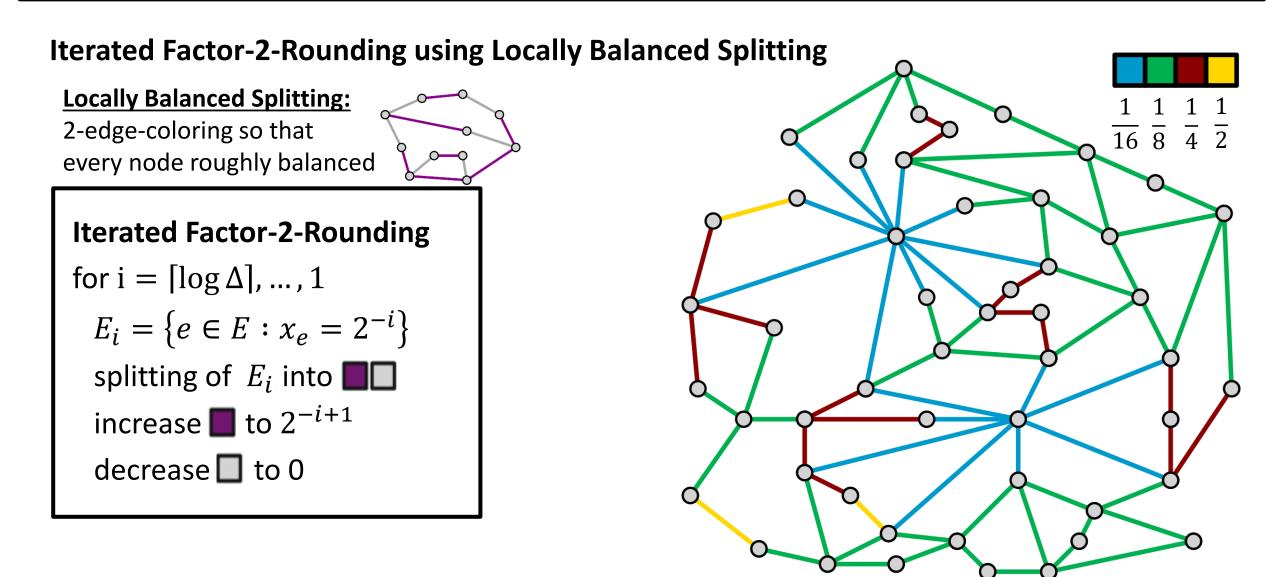
#### Iterated Factor-2-Rounding using Locally Balanced Splitting

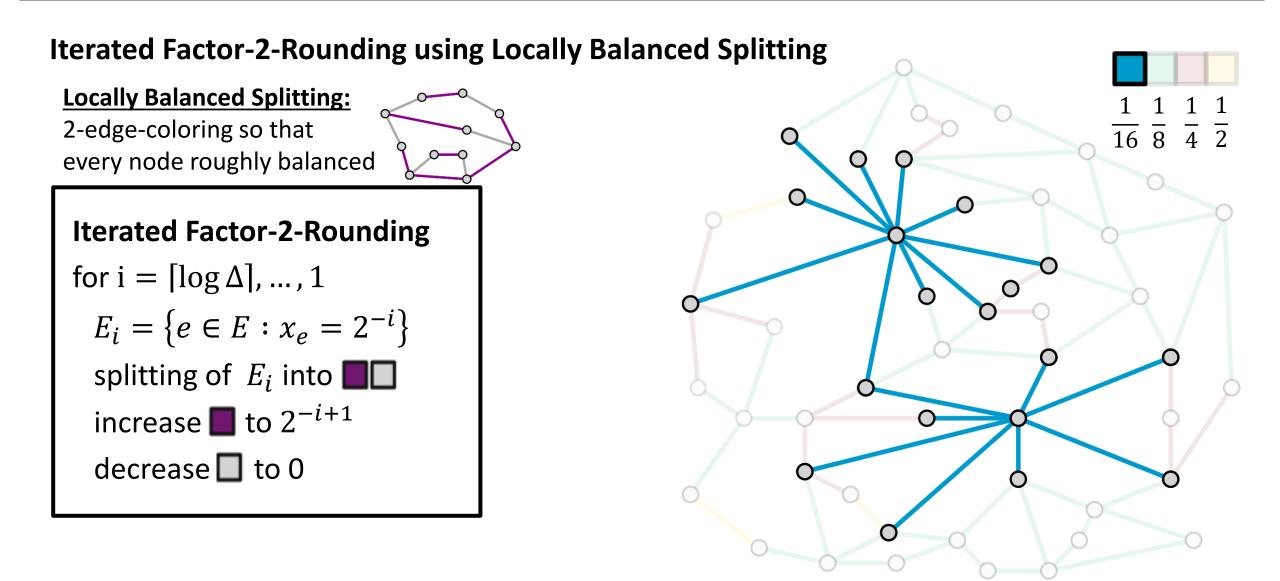
#### **Locally Balanced Splitting:**

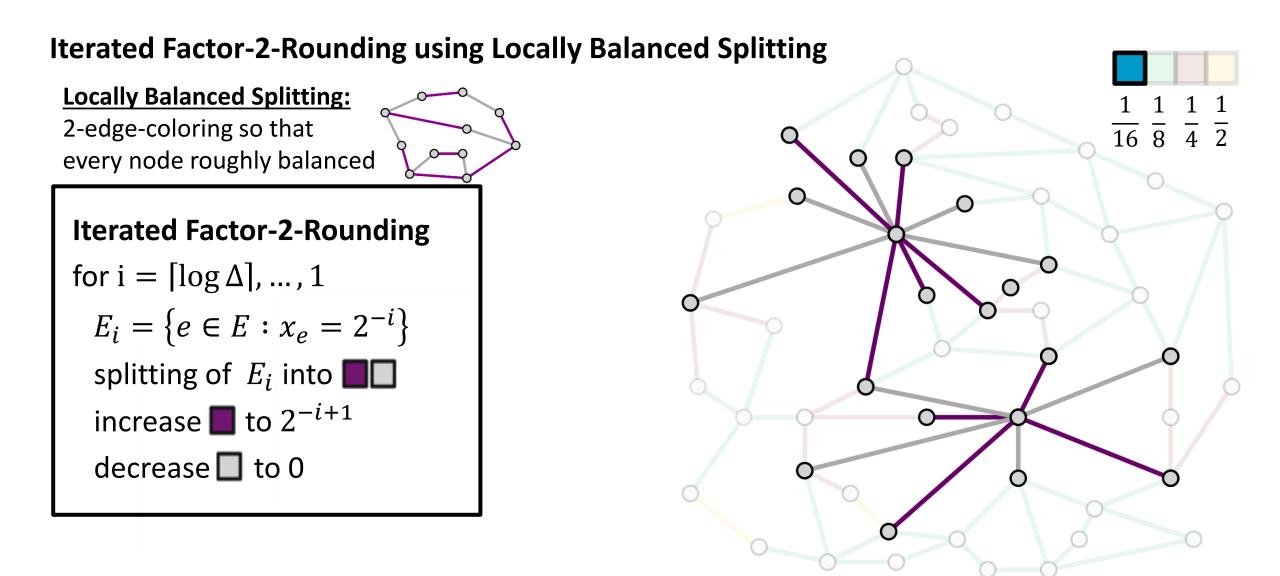
2-edge-coloring so that every node roughly balanced

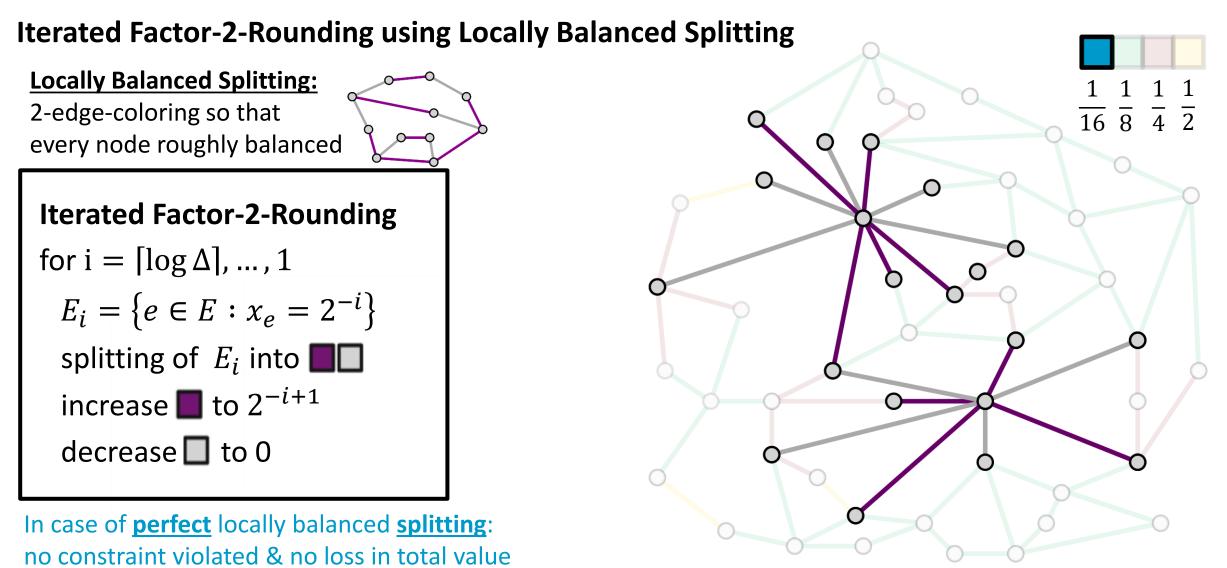


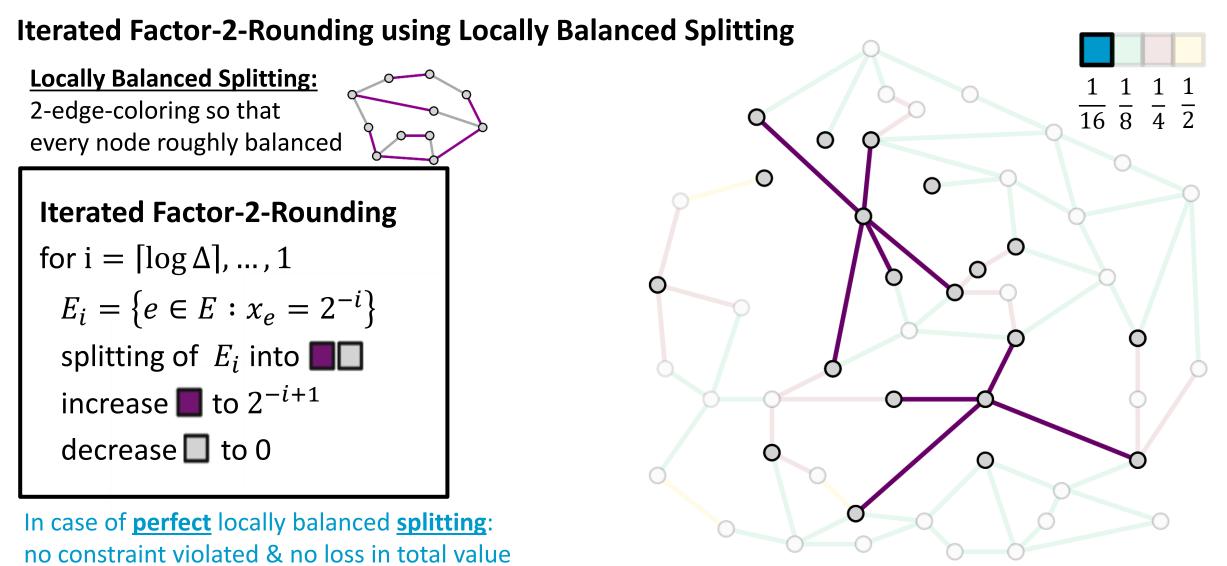


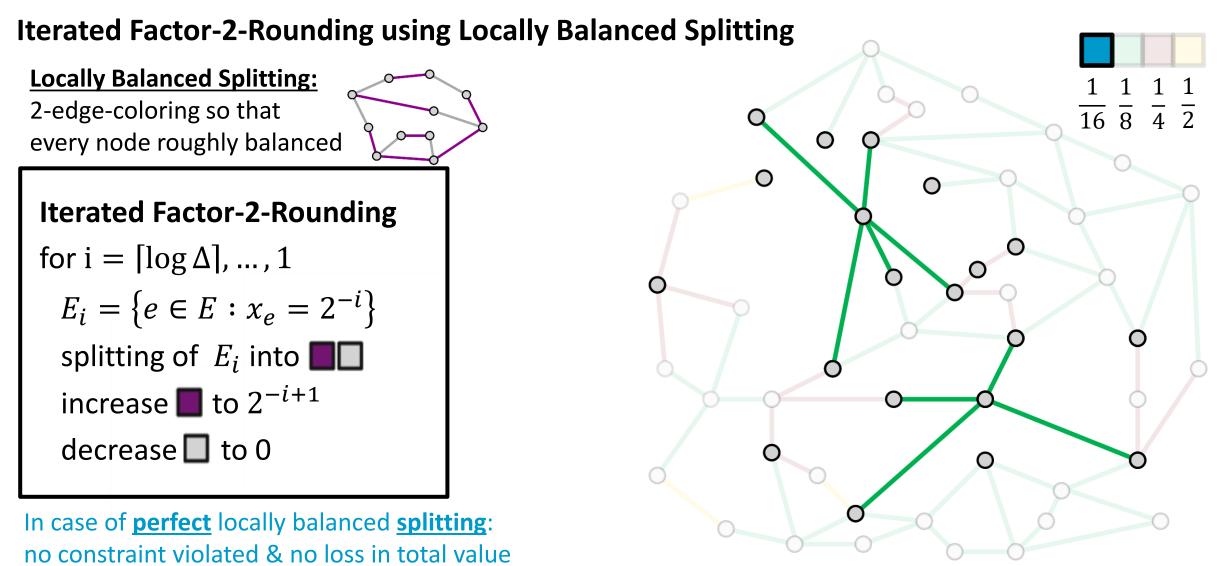


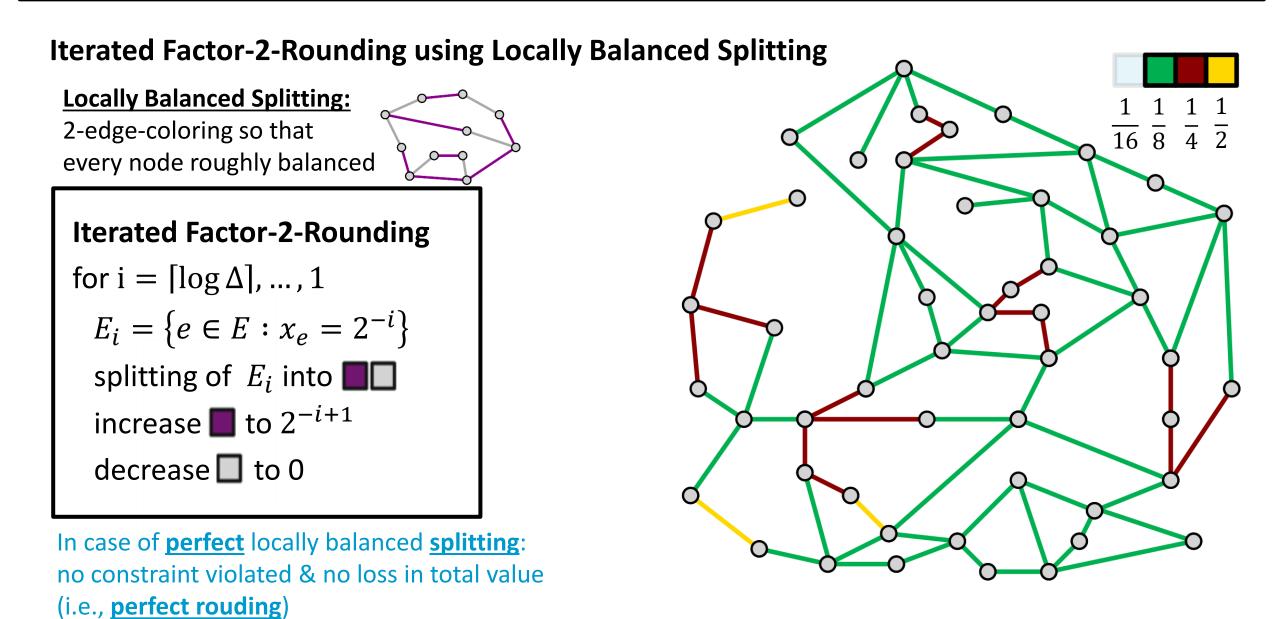


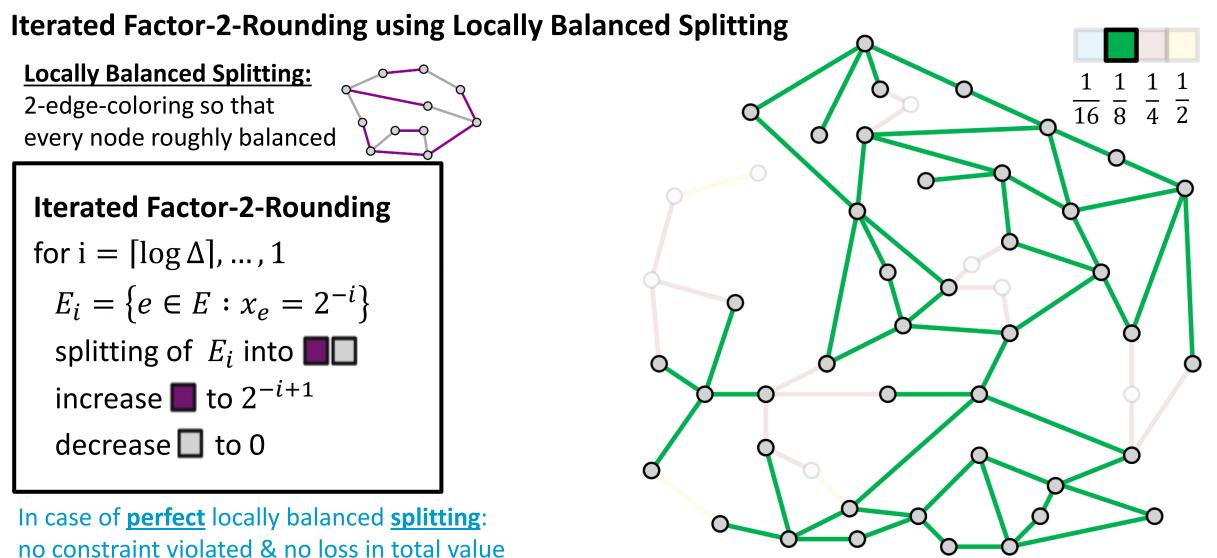


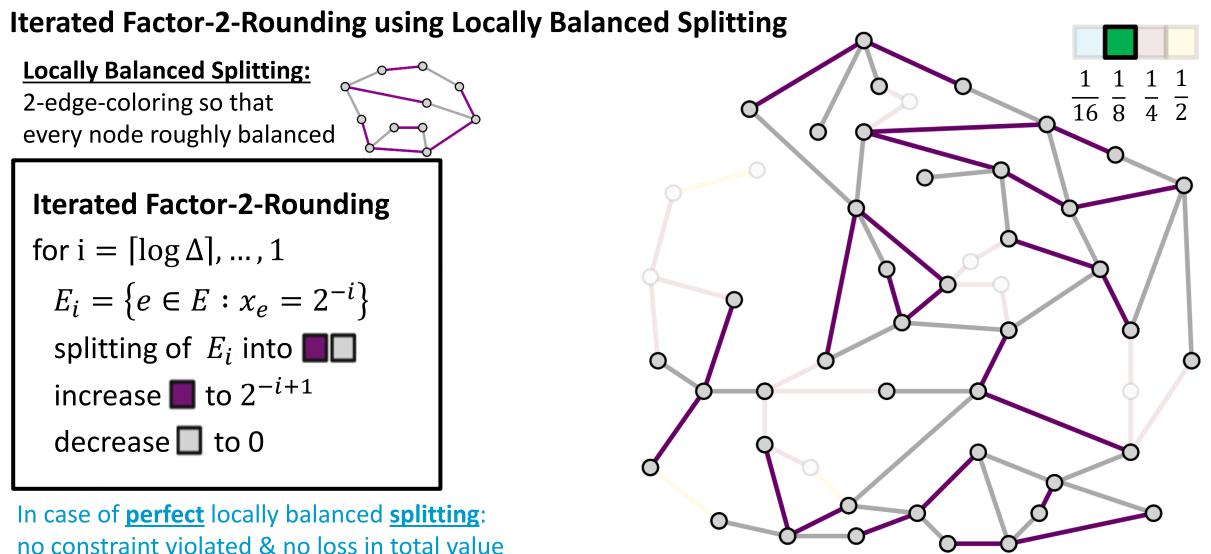


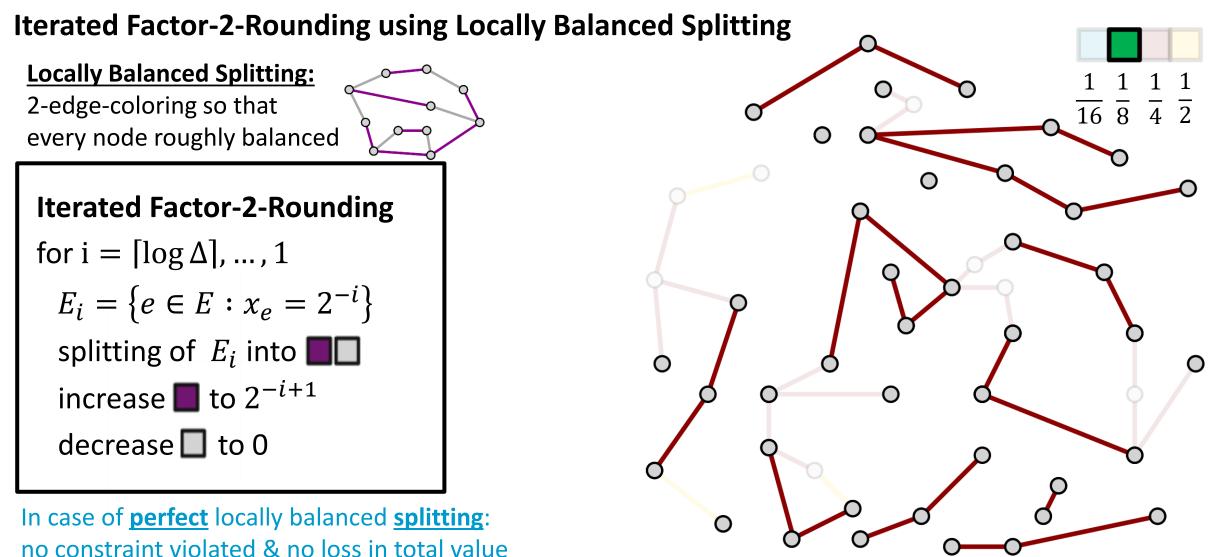


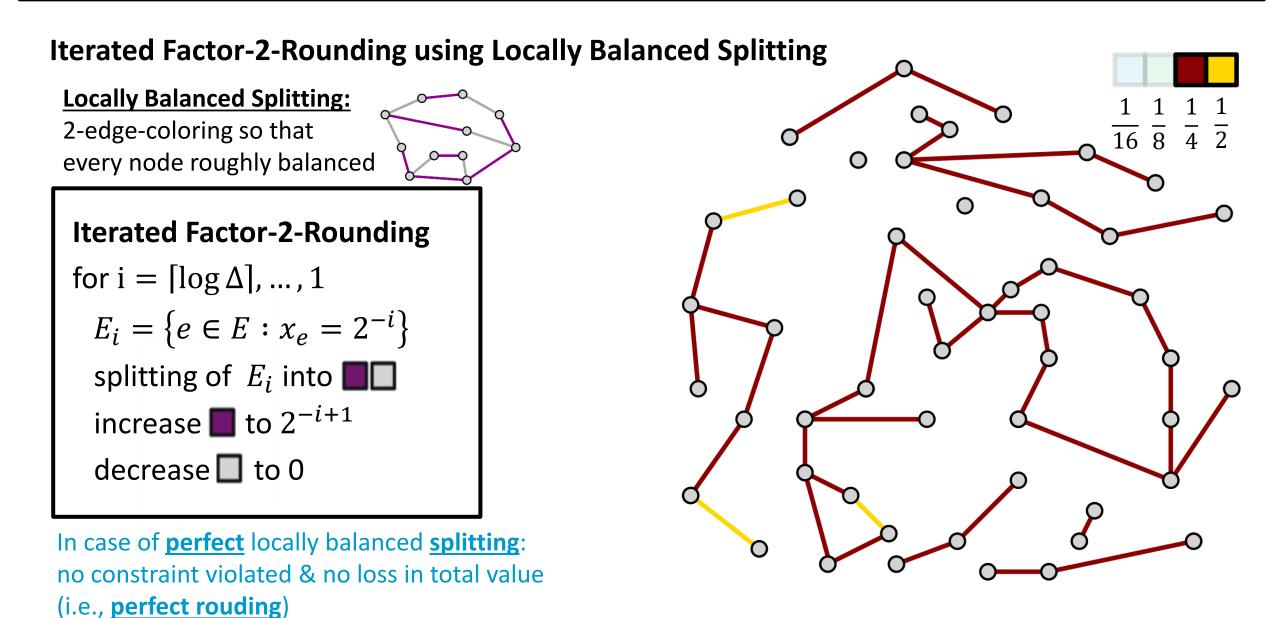


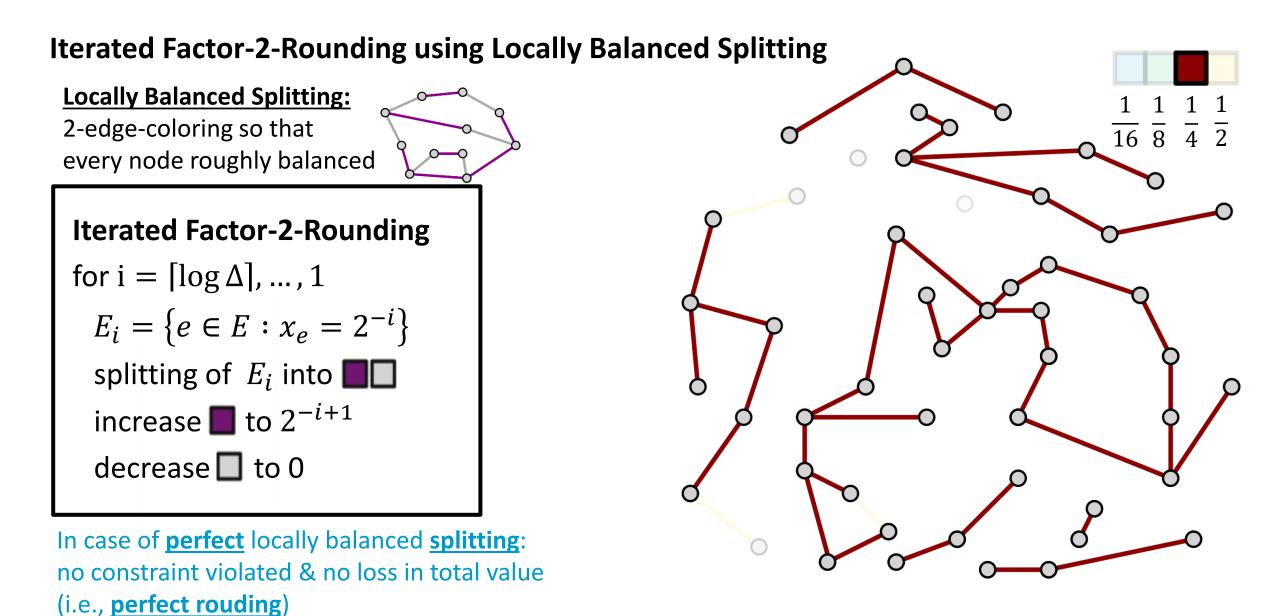


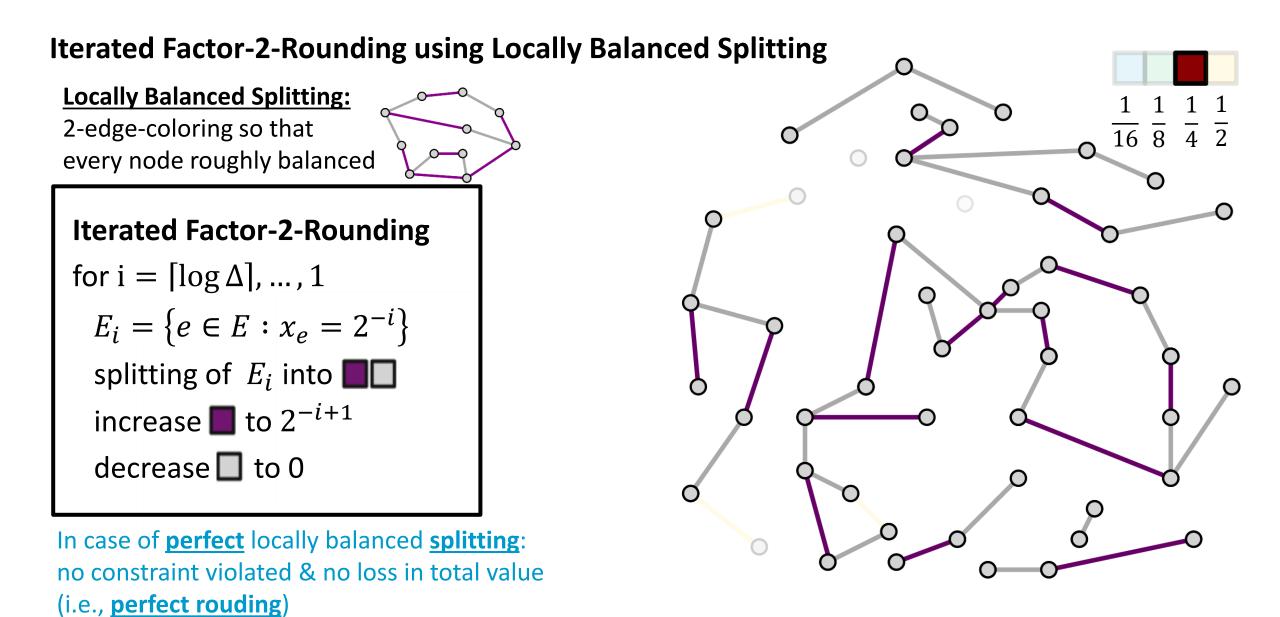


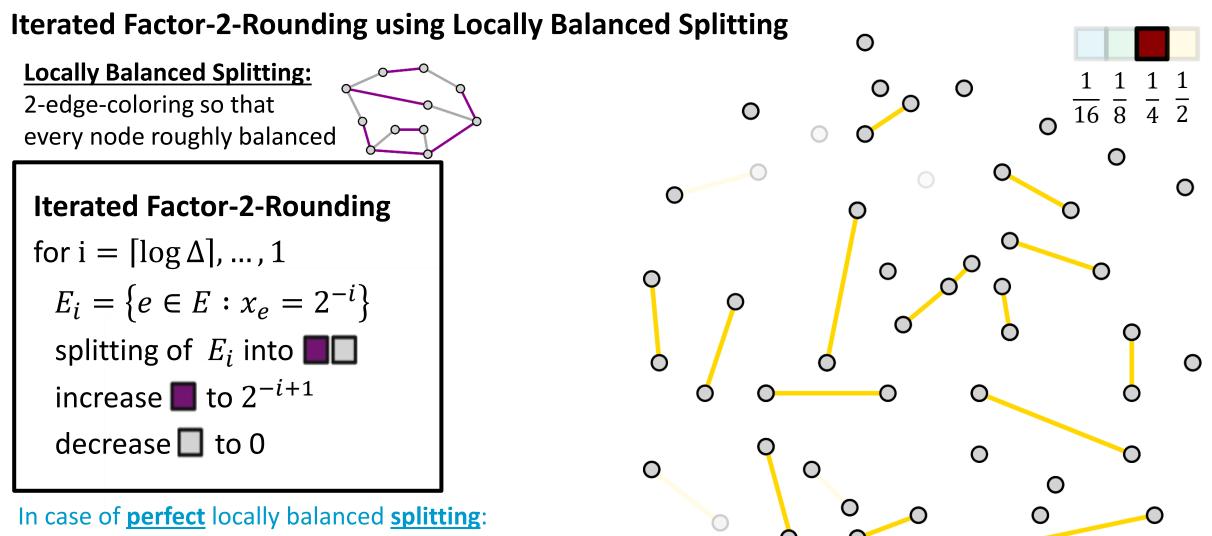




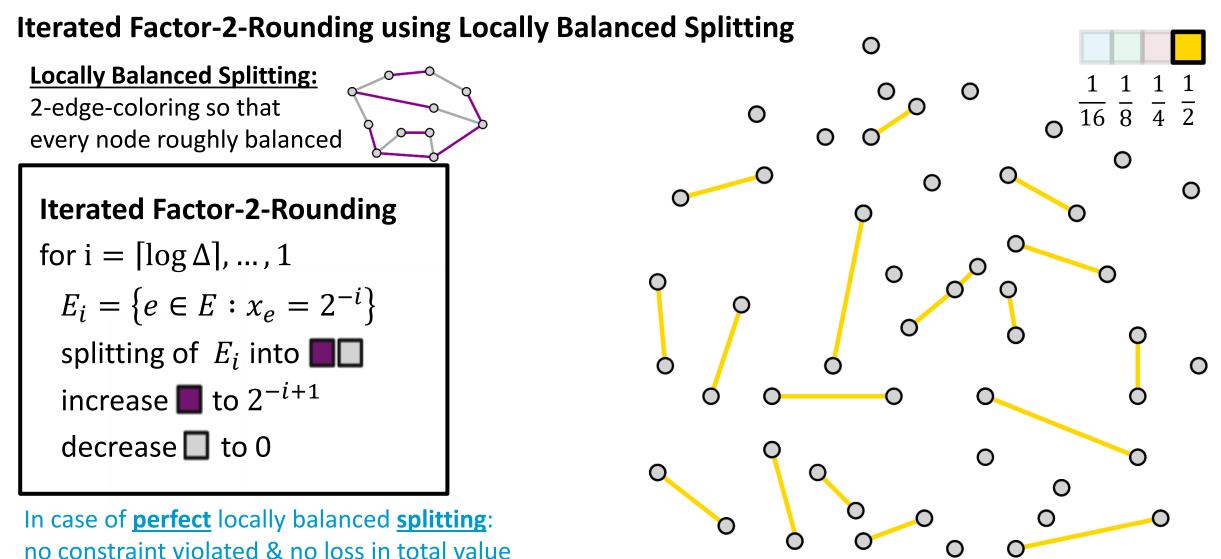




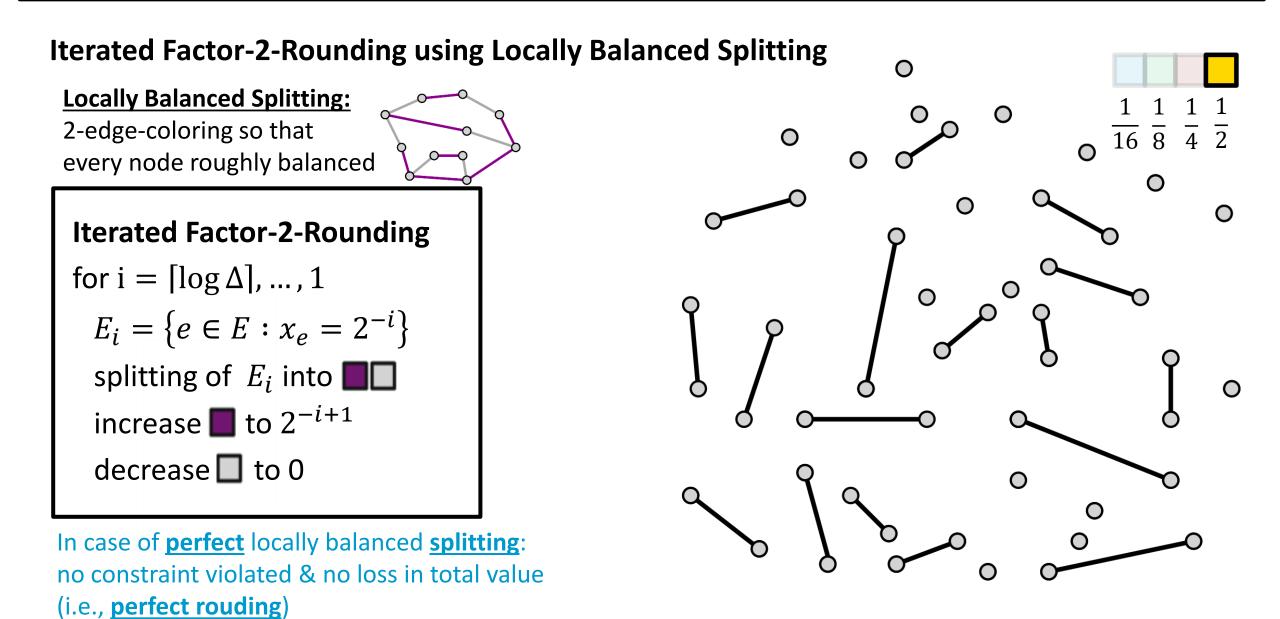




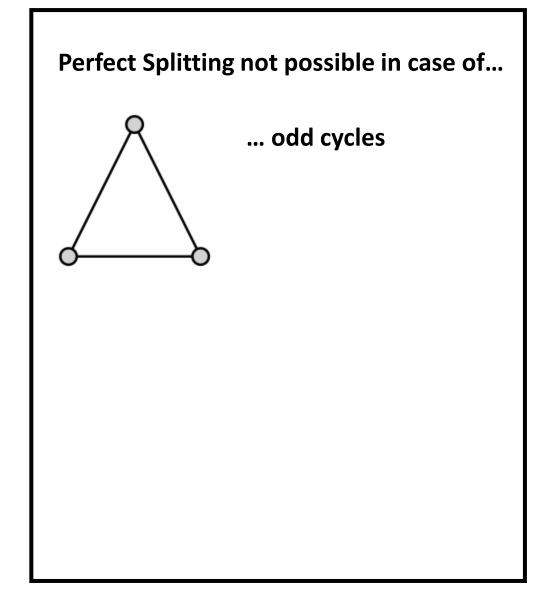
no constraint violated & no loss in total value (i.e., **perfect rouding**)

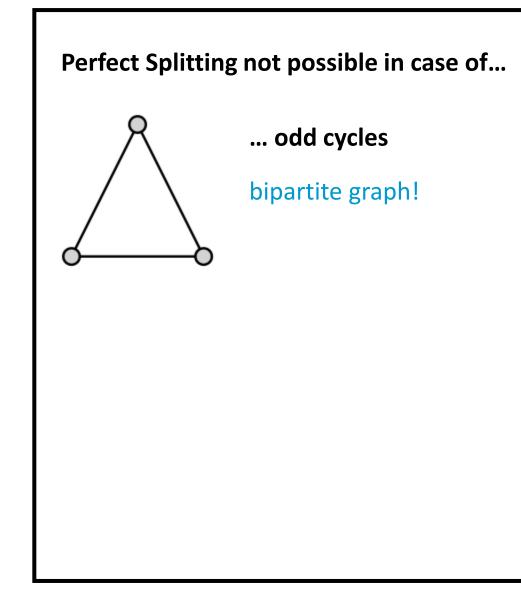


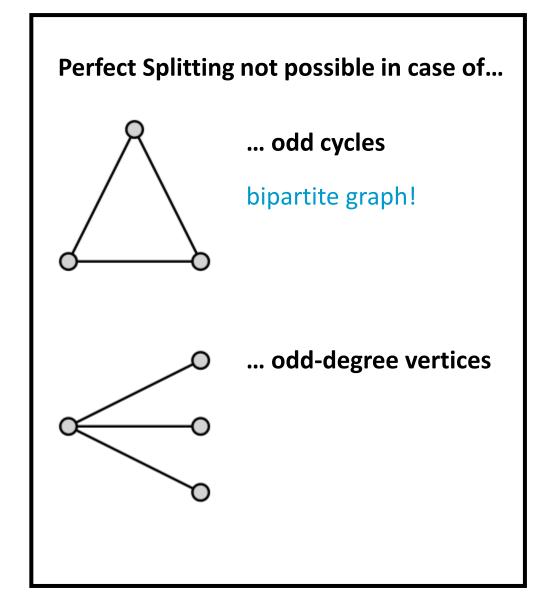
(i.e., **perfect rouding**)

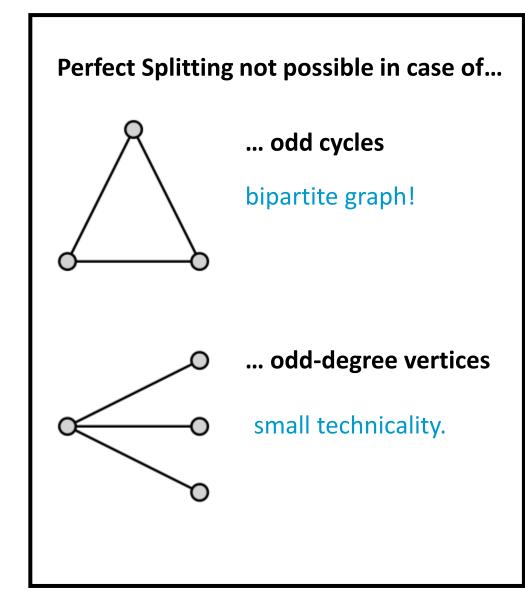


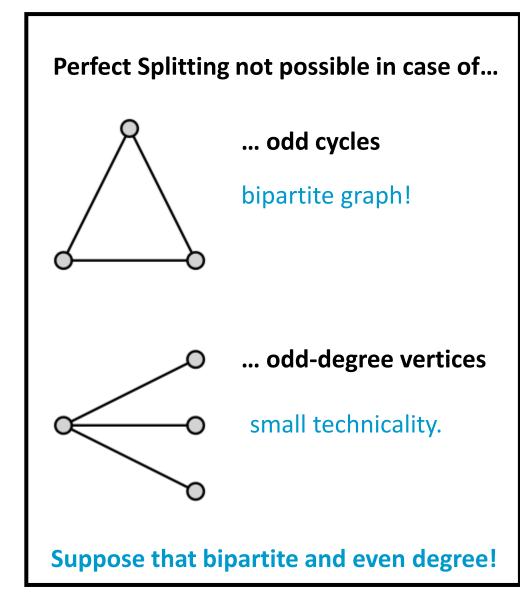
Perfect Splitting not possible in case of...









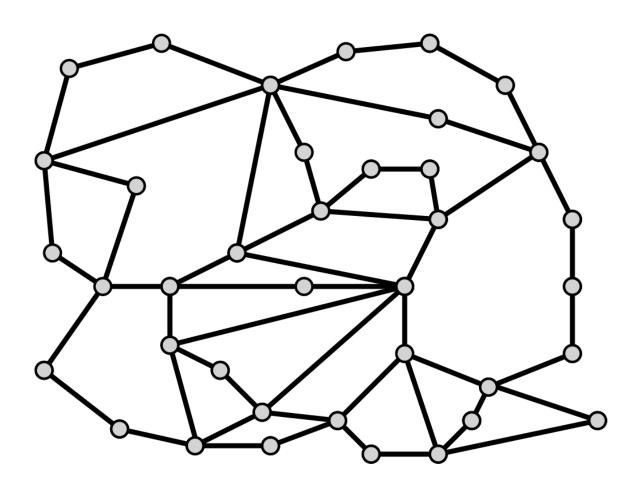


### Sequential Perfect Splitting\*

Repeat until all edges colored pick arbitrary cycle alternate

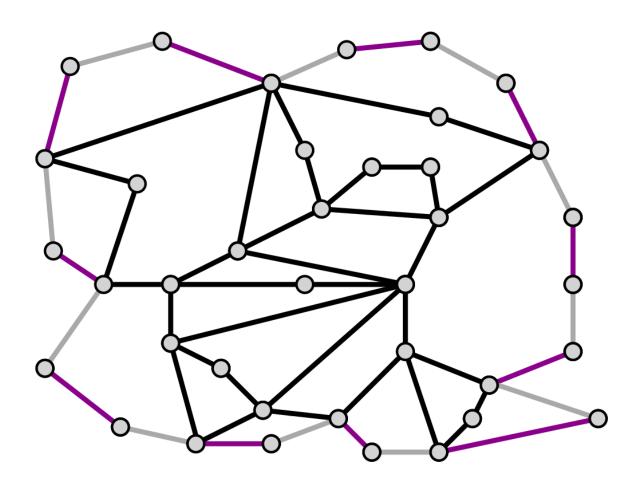
### Sequential Perfect Splitting\*

Repeat until all edges colored pick arbitrary cycle alternate



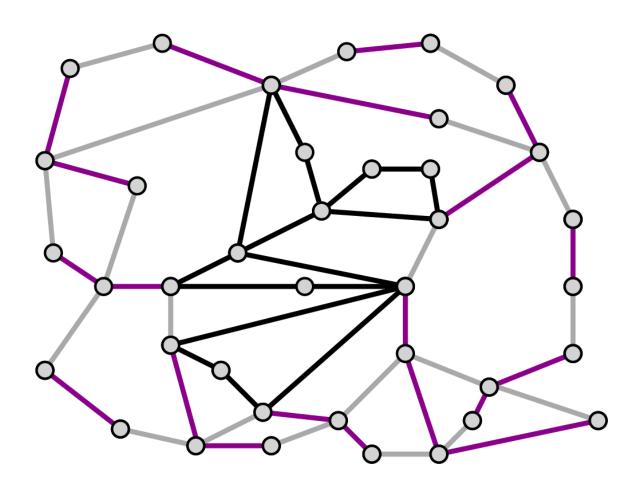
### Sequential Perfect Splitting\*

Repeat until all edges colored pick arbitrary cycle alternate



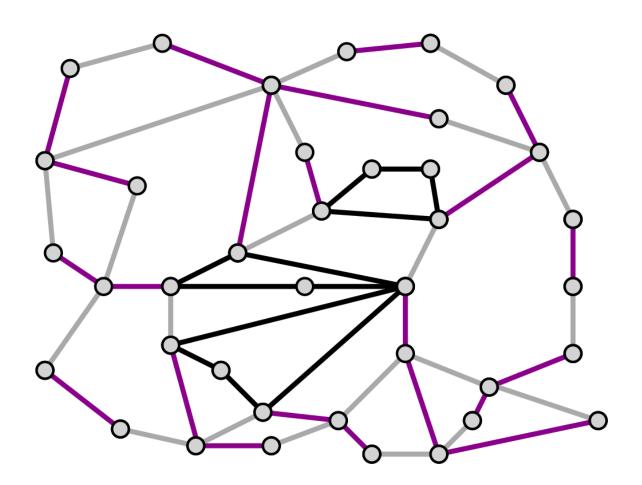
### Sequential Perfect Splitting\*

Repeat until all edges colored pick arbitrary cycle alternate



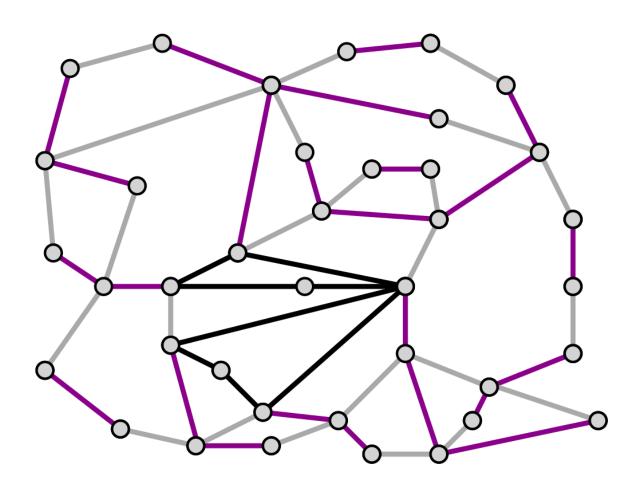
### Sequential Perfect Splitting\*

Repeat until all edges colored pick arbitrary cycle alternate



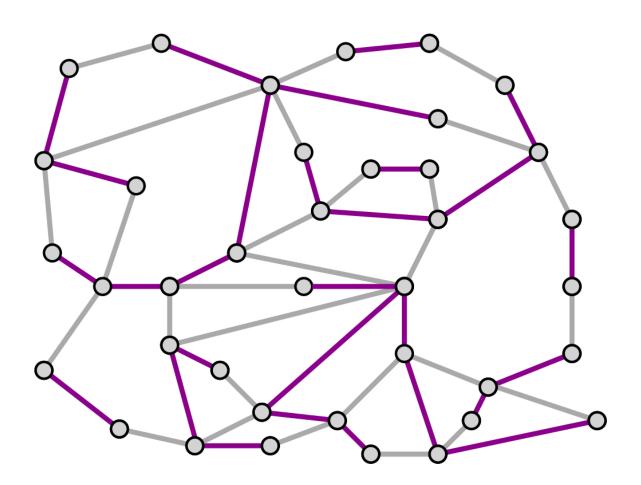
### Sequential Perfect Splitting\*

Repeat until all edges colored pick arbitrary cycle alternate



### Sequential Perfect Splitting\*

Repeat until all edges colored pick arbitrary cycle alternate



#### Sequential Perfect Splitting\*

Repeat until all edges colored pick arbitrary cycle



## LOCAL Almost-Perfect Splitting\*

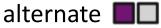
Decompose into edge-disjoint cycles In parallel, for all cycles

- A) **Short cycles** of length  $O(\log \Delta)$ 
  - alternate
- B) Long cycles
  - chop at length  $\Theta(\log \Delta)$
  - set boundary to 0

  - alternate **I** in between

### Sequential Perfect Splitting\*

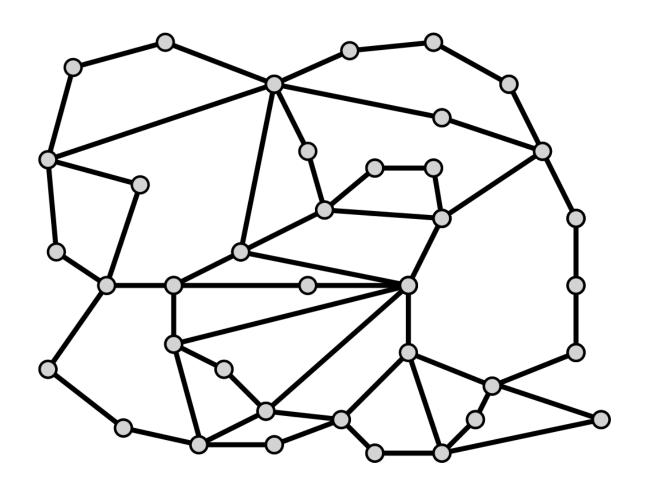
Repeat until all edges colored pick arbitrary cycle



## LOCAL Almost-Perfect Splitting\*

Decompose into edge-disjoint cycles In parallel, for all cycles

- A) **Short cycles** of length  $O(\log \Delta)$ 
  - alternate
- B) Long cycles
  - chop at length  $\Theta(\log \Delta)$
  - set boundary to 0
  - alternate **E** in between



#### Sequential Perfect Splitting\*

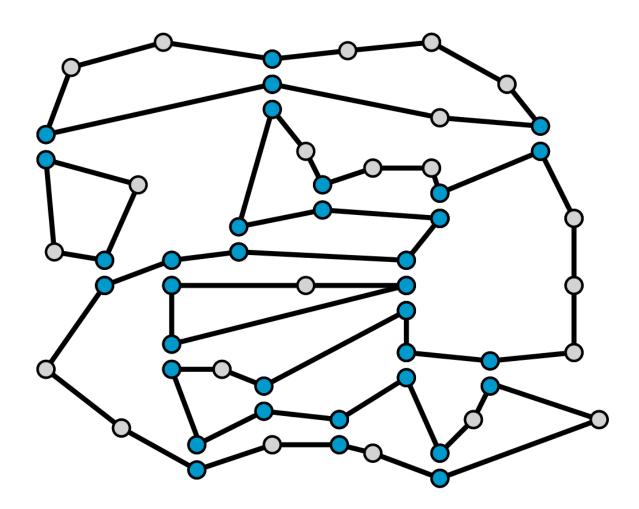
Repeat until all edges colored pick arbitrary cycle



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  - alternate
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  - chop at length  $\Theta(\log \Delta)$
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### Sequential Perfect Splitting\*

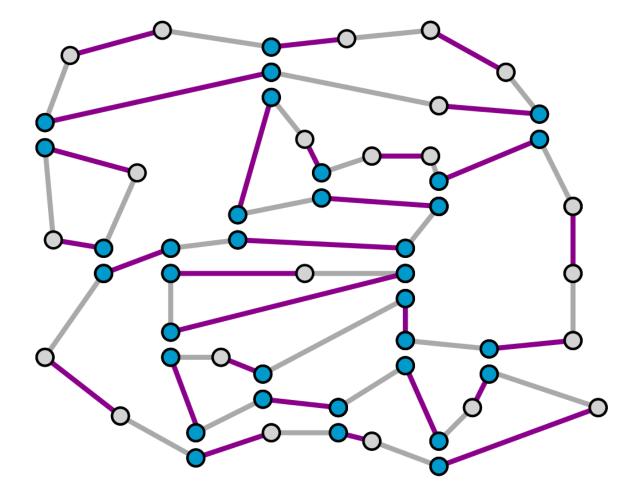
Repeat until all edges colored pick arbitrary cycle

alternate 🔳

## LOCAL Almost-Perfect Splitting\*

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  - alternate
- B) Long cycles
  - chop at length  $\Theta(\log \Delta)$
  - set boundary to 0
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#### Sequential Perfect Splitting\*

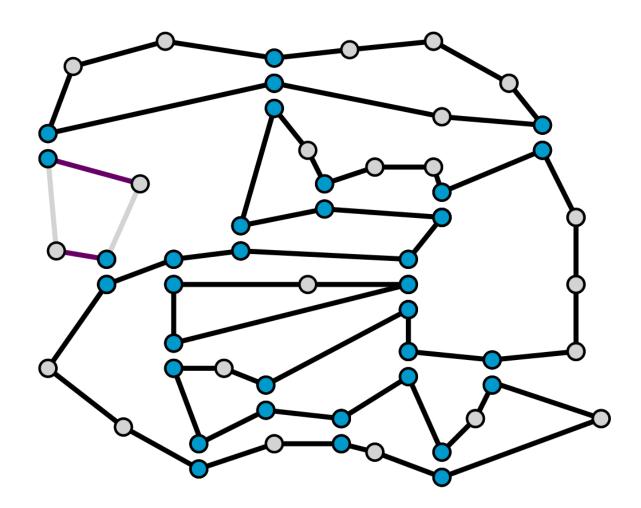
Repeat until all edges colored pick arbitrary cycle



### LOCAL Almost-Perfect Splitting\*

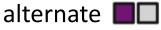
Decompose into edge-disjoint cycles In parallel, for all cycles

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- B) Long cycles
  - chop at length  $\Theta(\log \Delta)$
  - set boundary to 0
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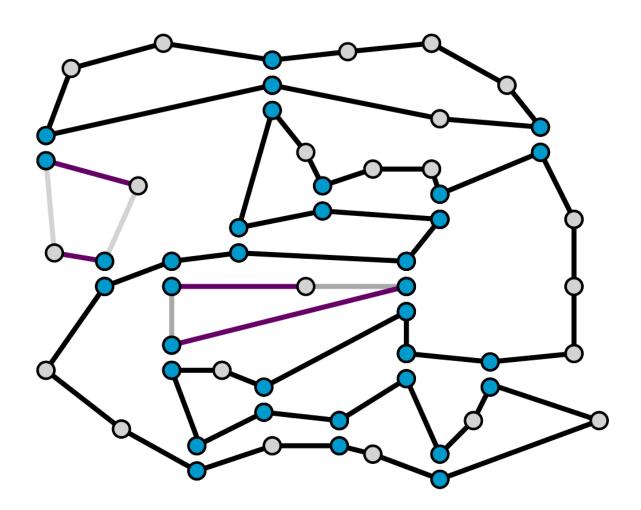
Repeat until all edges colored pick arbitrary cycle



## LOCAL Almost-Perfect Splitting\*

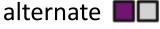
Decompose into edge-disjoint cycles In parallel, for all cycles

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- B) Long cycles
  - chop at length  $\Theta(\log \Delta)$
  - set boundary to 0
  - alternate **E** in between



#### Sequential Perfect Splitting\*

Repeat until all edges colored pick arbitrary cycle

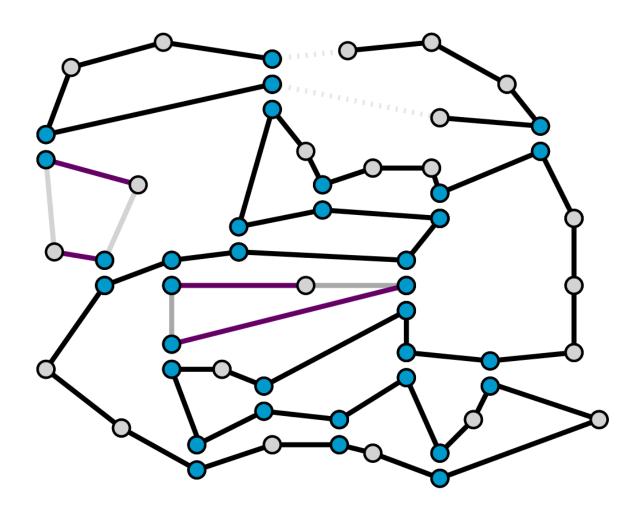


### LOCAL Almost-Perfect Splitting\*

Decompose into edge-disjoint cycles In parallel, for all cycles

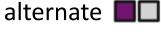
- A) **Short cycles** of length  $O(\log \Delta)$ 
  - alternate
- B) Long cycles
  - chop at length  $\Theta(\log \Delta)$
  - set boundary to 0

  - alternate **E** in between



### Sequential Perfect Splitting\*

Repeat until all edges colored pick arbitrary cycle

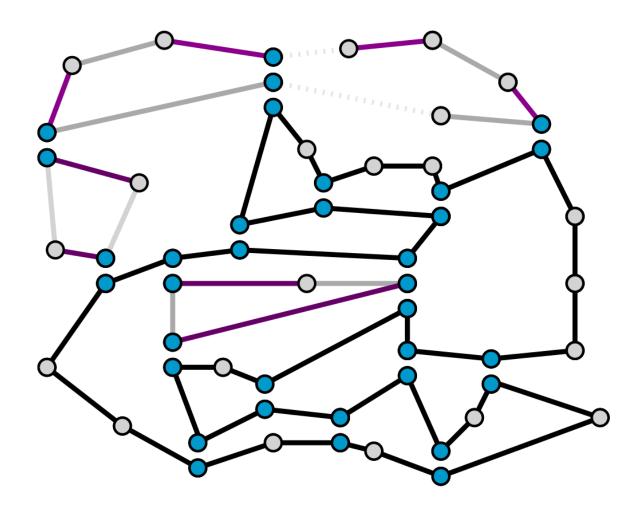


### LOCAL Almost-Perfect Splitting\*

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  - alternate
- B) Long cycles
  - chop at length  $\Theta(\log \Delta)$
  - set boundary to 0

  - alternate **E** in between



### Sequential Perfect Splitting\*

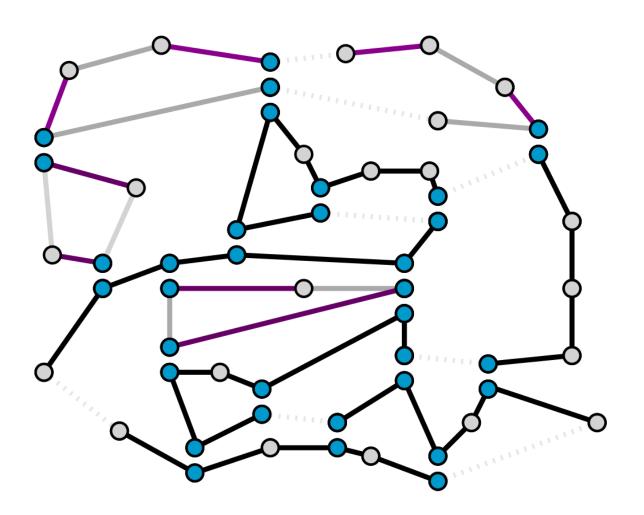
Repeat until all edges colored pick arbitrary cycle

alternate

## LOCAL Almost-Perfect Splitting\*

Decompose into edge-disjoint cycles In parallel, for all cycles

- A) **Short cycles** of length  $O(\log \Delta)$ 
  - alternate
- B) Long cycles
  - chop at length  $\Theta(\log \Delta)$
  - set boundary to 0
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### Sequential Perfect Splitting\*

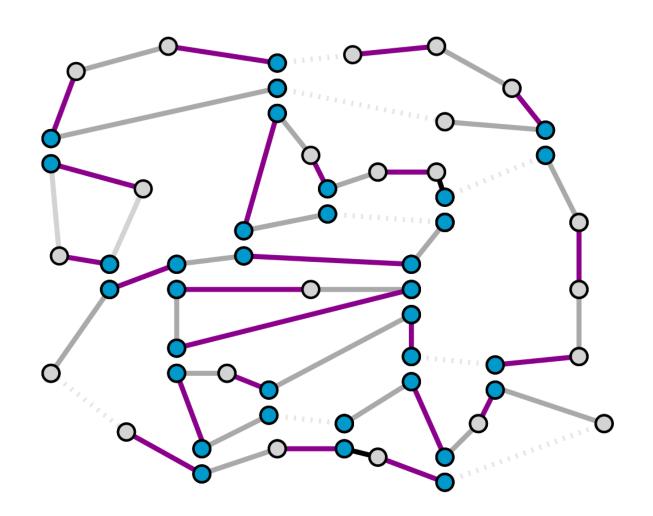
Repeat until all edges colored pick arbitrary cycle

alternate

### LOCAL Almost-Perfect Splitting\*

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  - alternate
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  - chop at length  $\Theta(\log \Delta)$
  - set boundary to 0
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### Sequential Perfect Splitting\*

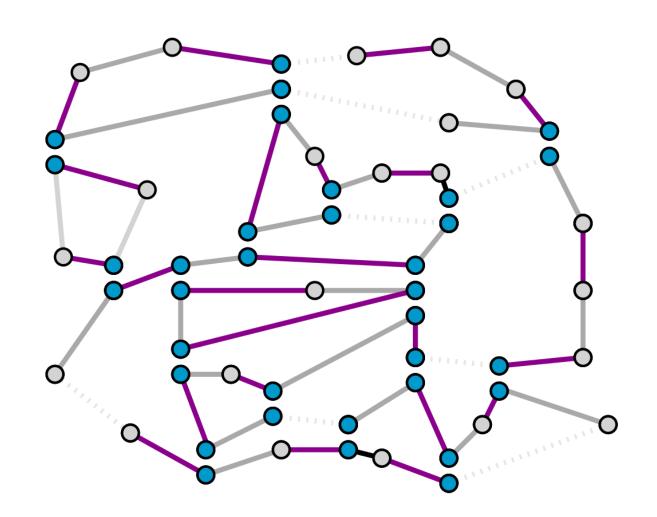
Repeat until all edges colored pick arbitrary cycle alternate

### LOCAL Almost-Perfect Splitting\*

Decompose into edge-disjoint cycles In parallel, for all cycles

- A) **<u>Short cycles</u>** of length  $O(\log \Delta)$ 
  - alternate
- B) Long cycles

chop at length  $\Theta(\log \Delta)$ set boundary to 0  $\Theta\left(\frac{1}{\log \Delta}\right)$  loss alternate  $\square$  in between



### Sequential Perfect Splitting\*

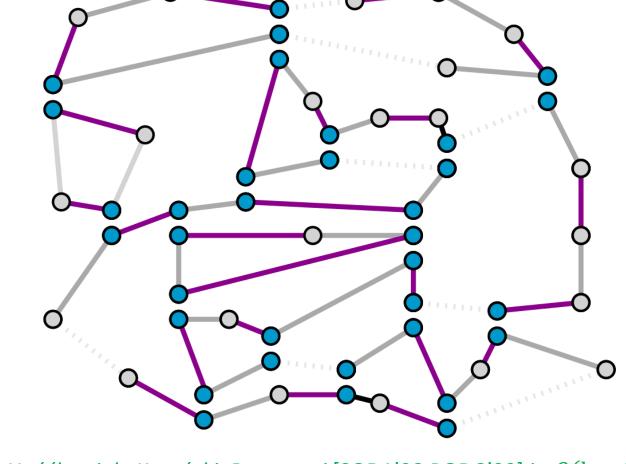
Repeat until all edges colored pick arbitrary cycle alternate

### LOCAL Almost-Perfect Splitting\*

Decompose into edge-disjoint cycles In parallel, for all cycles

- A) Short cycles of length  $O(\log \Delta)$ 
  - alternate
- B) <u>Long cycles</u>

chop<sup>\*</sup>at length  $\Theta(\log \Delta)$ set boundary to 0  $\Theta\left(\frac{1}{\log \Delta}\right)$  loss alternate  $\square$  in between



\* by Hańćkowiak, Karoński, Panconesi [SODA'98,PODC'99] in  $O(\log \Delta)$ 

### Sequential Perfect Splitting\*

Repeat until all edges colored pick arbitrary cycle alternate

### LOCAL Almost-Perfect Splitting\*

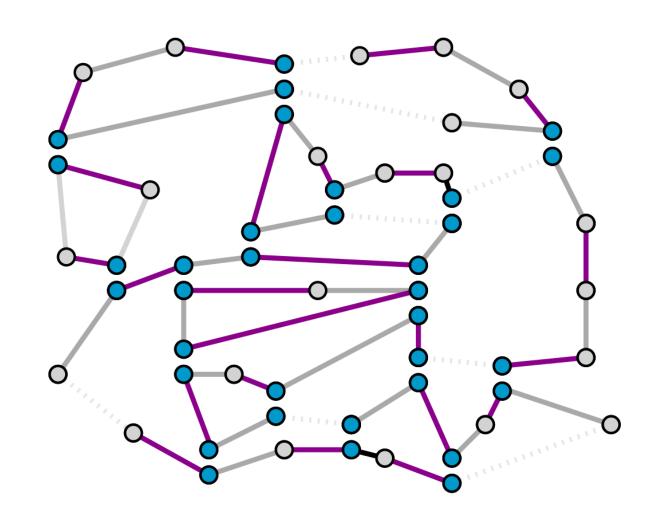
Decompose into edge-disjoint cycles In parallel, for all cycles

A) **<u>Short cycles</u>** of length  $O(\log \Delta)$ 

alternate



chop\*at length  $\Theta(\log \Delta)$ set boundary to 0  $\Theta\left(\frac{1}{\log \Delta}\right)$  loss alternate in between



\* by Hańćkowiak, Karoński, Panconesi [SODA'98,PODC'99] in  $O(\log \Delta)$ 

#### Over all $O(\log \Delta)$ rounding iterations, total loss still constant!

# I) 4-Approximate Fractional Matching

 $O(\log \Delta)$  rounds

## **II)** Rounding Fractional Bipartite Matching

 $O(\log^2 \Delta)$  rounds, O(1) loss

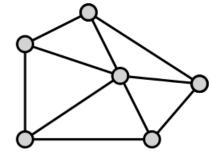
 $O(\log^2 \Delta)$  rounds



 $O(\log^2 \Delta)$  rounds

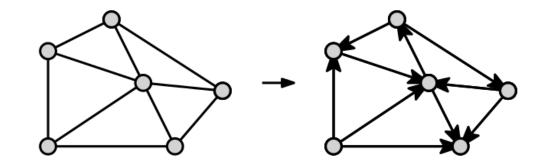


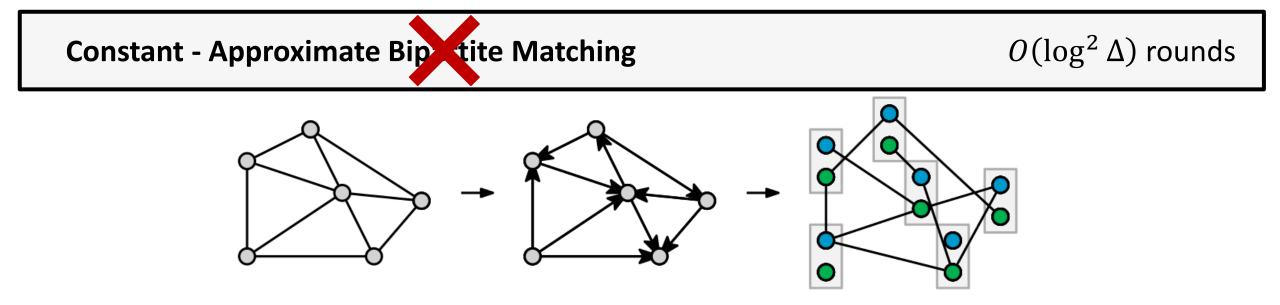
 $O(\log^2 \Delta)$  rounds

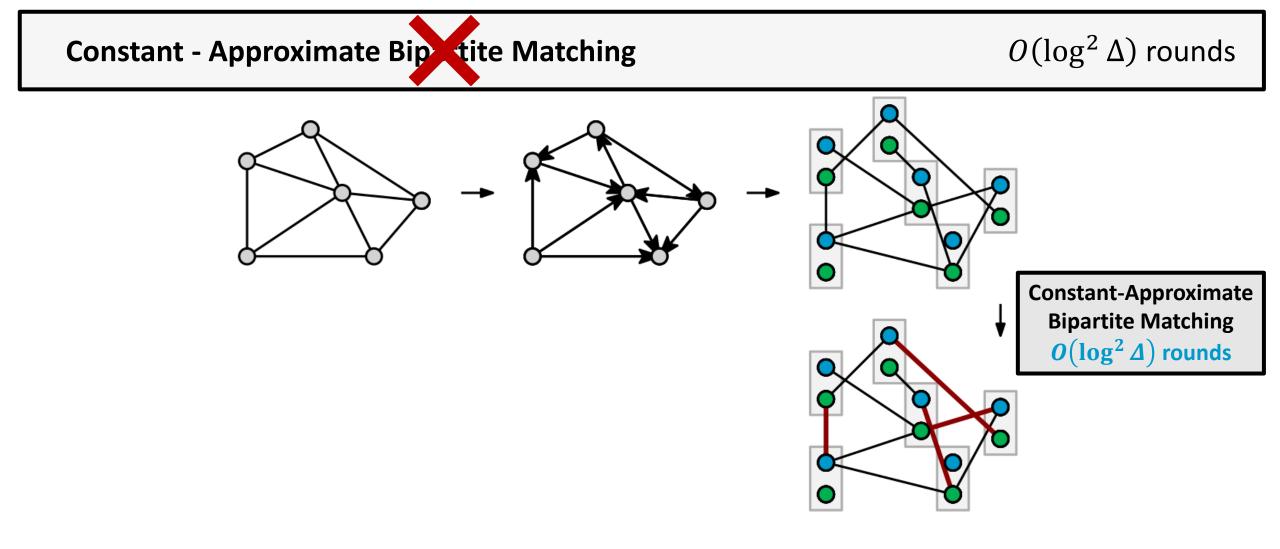


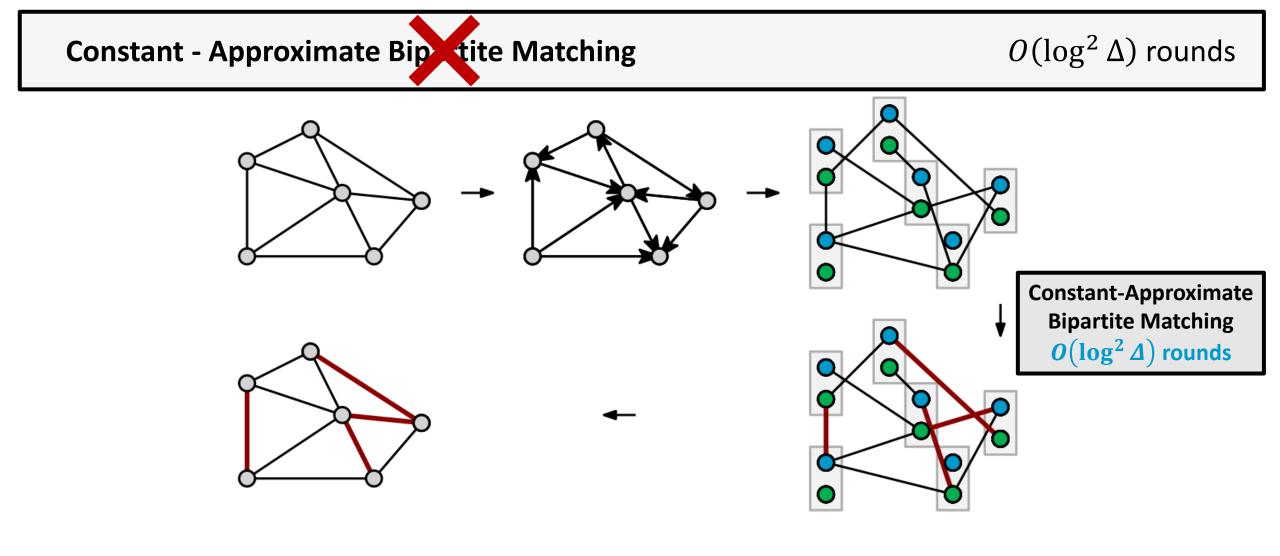


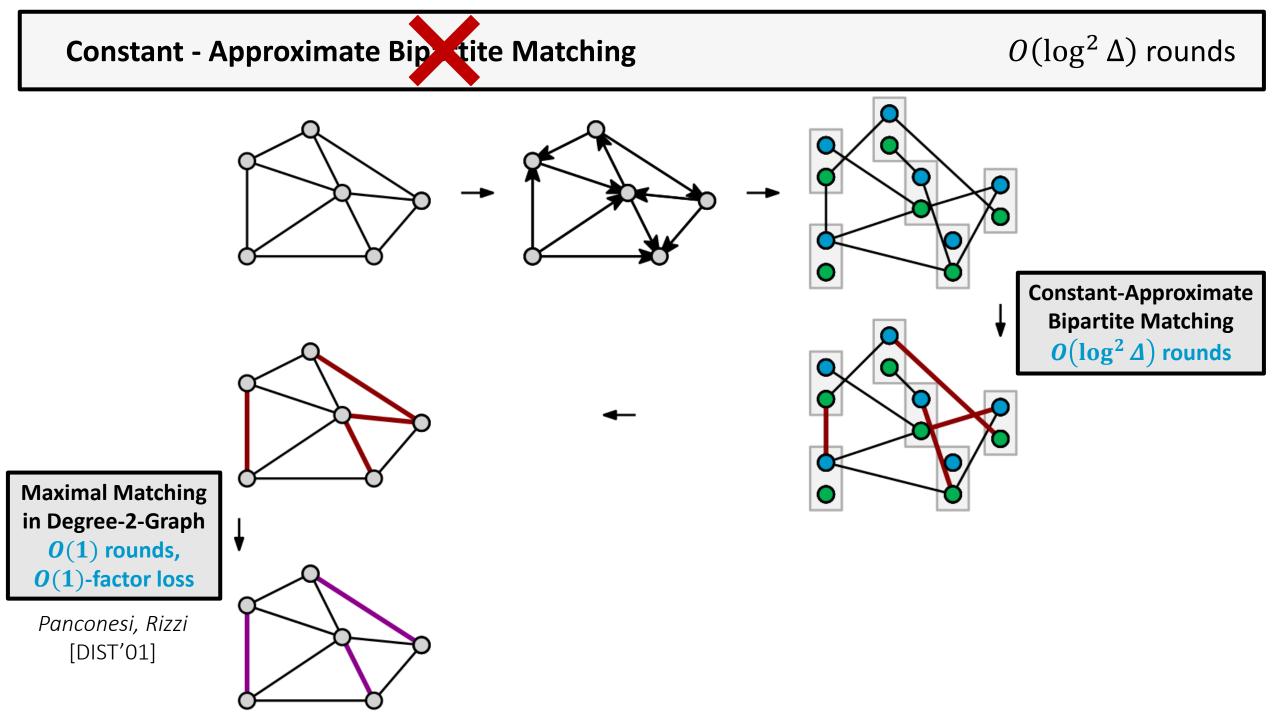
 $O(\log^2 \Delta)$  rounds











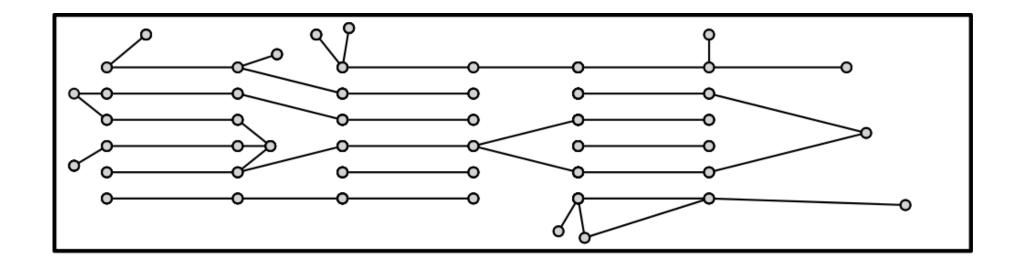
Maximal

 $O(\log^2 \Delta \cdot \log n)$ 

 $O(\log^2 \Delta)$  rounds

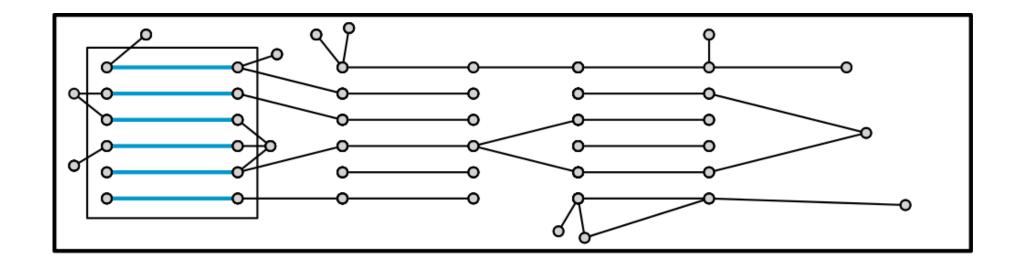
 $O(\log^2 \Delta)$  rounds

#### Maximal



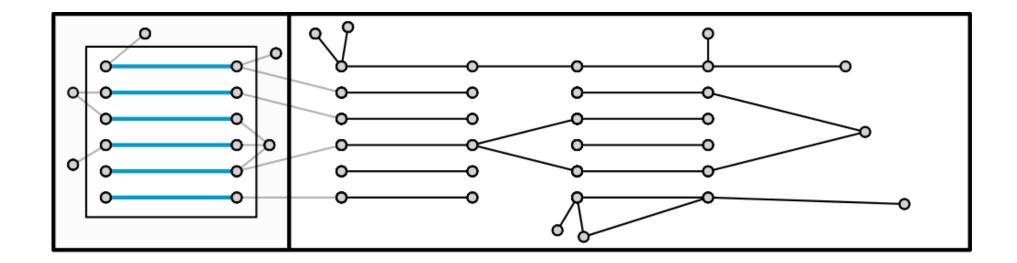
 $O(\log^2 \Delta)$  rounds

#### Maximal



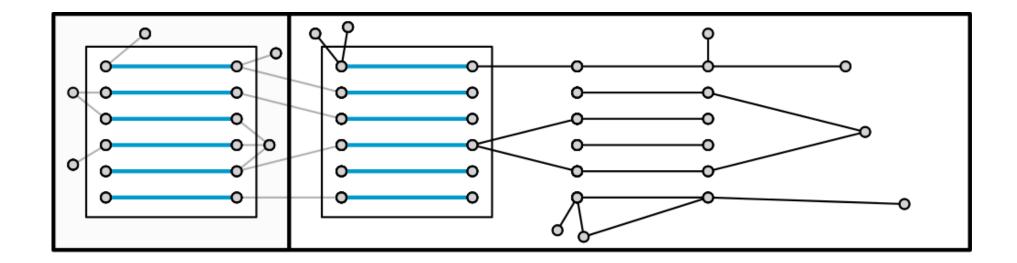
**Maximal** 

 $O(\log^2 \Delta)$  rounds



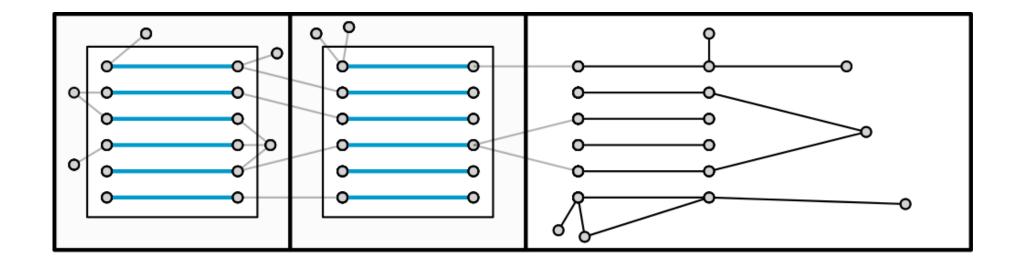
**Maximal** 

 $O(\log^2 \Delta)$  rounds



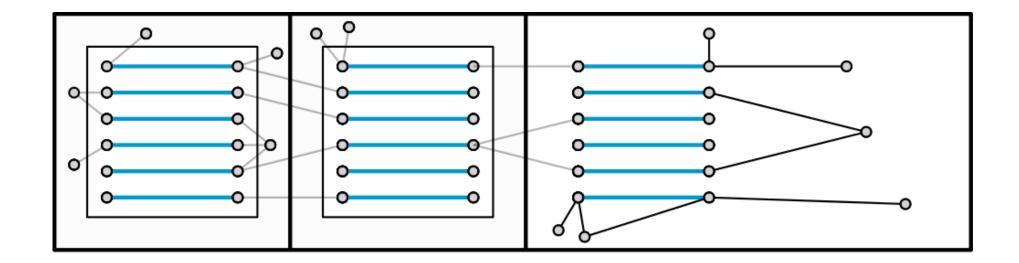
**Maximal** 

 $O(\log^2 \Delta)$  rounds



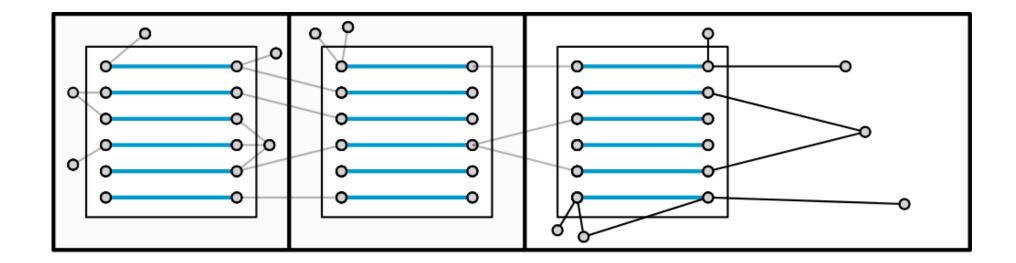
**Maximal** 

 $O(\log^2 \Delta)$  rounds



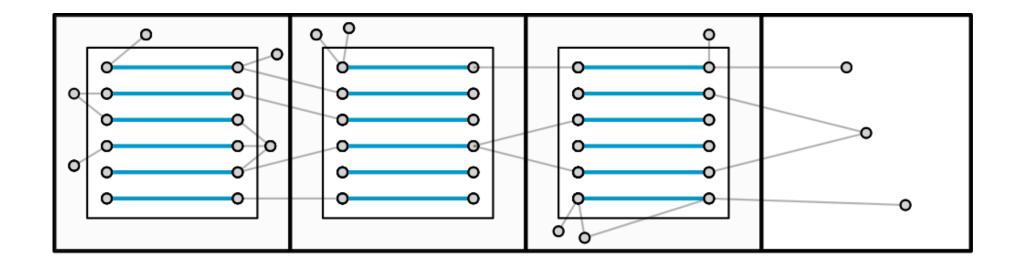
**Maximal** 

 $O(\log^2 \Delta)$  rounds



**Maximal** 

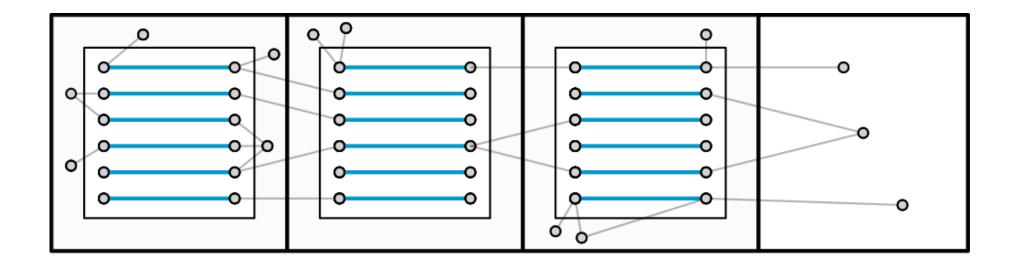
 $O(\log^2 \Delta)$  rounds



Maximal

 $O(\log^2 \Delta)$  rounds

#### $O(\log^2 \Delta \cdot \log n)$

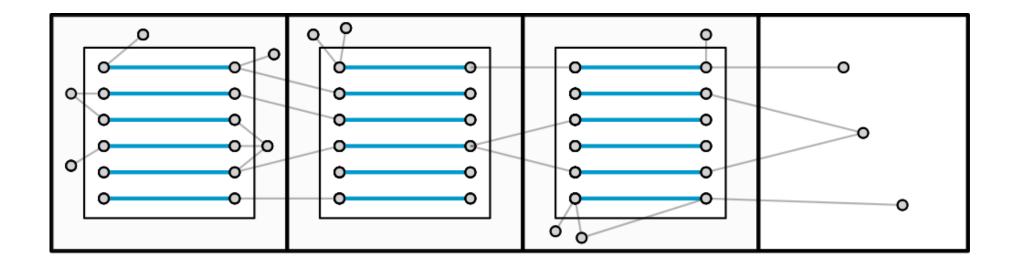


#### maximum matching size in remainder graph decreases by constant factor

Maximal

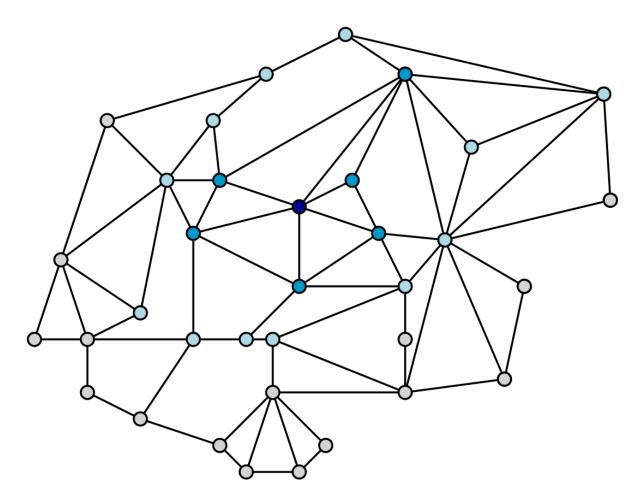
 $O(\log^2 \Delta)$  rounds

$$O(\log^2 \Delta \cdot \log n)$$



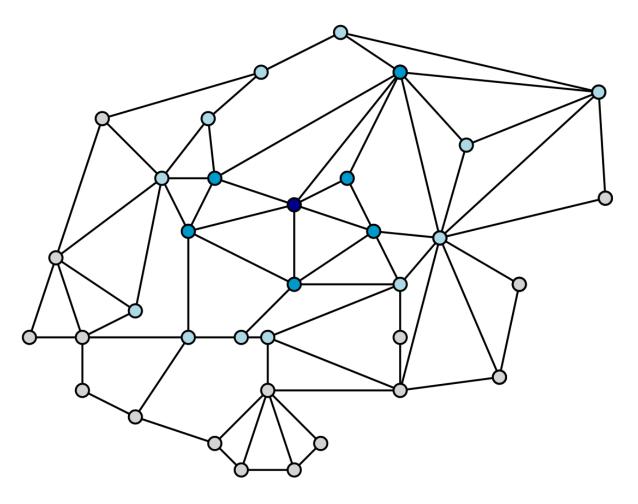
maximum matching size in remainder graph decreases by constant factor

after  $O(\log n)$  iterations, maximum matching size is 0, hence graph empty



**<u>Open Question:</u>**  $O(\log \Delta \cdot \log n)$ ?

What is Locality of Maximal Matching?



#### Thank you!

#### **<u>Open Question:</u>** $O(\log \Delta \cdot \log n)$ ?

What is Locality of Maximal Matching?