1 Bad Queues in a Mesh

In order to obtain big queues at a node \(v\), packets need to arrive from all three possible directions in each step. Therefore, the maximum number of destinations from one direction in a column is \(m-2\). See Figure 1. In each step, the queue grows by 2 and there are \((m-2)/3\) steps. Thus the queue size grows to

\[
\frac{2}{3}(m - 2)
\]

2 Good Queues in a Mesh

Following the lead given in the exercise we want to bound the probability \(P_{2em}\) that a particular column contains \(2em\) or more destination packets. Analogous to the proof of Theorem 4.10 in the lecture, we have

\[
P_{2em} \leq \left( \frac{m^2}{2em} \right) \cdot \left( \frac{1}{m} \right)^{2em}
\]  

(1)

(since we put \(2em\) out of the \(m^2\) destination packets in that column, each with a probability \(1/m\)). Using the inequality of the lecture (in the same proof) we can further simplify this to

\[
P_{2em} \leq \left( \frac{em^2}{2em} \right)^{2em} \cdot \left( \frac{1}{m} \right)^{2em} = \left( \frac{1}{2} \right)^{2em}
\]

(2)

to obtain that the probability for a single column to contain more than \(2em\) packets is “really small” (i.e. in \(o(2^{-m})\)).
Since we want a bound on the column with the maximum number of destination packets, we can compute the probability $P_{2em}^{\text{all}}$ that all $m$ columns contain $2em$ or more packets:

$$P_{2em}^{\text{all}} \leq \sum_{i=1}^{m} P_{2em} = mP_{2em}$$  \hspace{1cm} (3)

since the probability for all columns is the union of the probabilities that in each column there are more than $2em$ packets. The union of probabilities is upper bounded by their sum. Plugging (2) into (3) we get that

$$P_{2em}^{\text{all}} \leq \frac{m}{2^{2em}} \leq \frac{1}{m^2}$$  \hspace{1cm} (4)

where we used that $m/2^m \leq 1/m^2$ for large $m$ since an exponential function grows faster than any polynomial.

 Altogether, the argument is then as follows: The probability that all columns contain less than $O(m)$ packets is high, namely in $1 - O(1/m^2)$. Therefore, we also have a high probability that the column containing the most number of destinations also gets only $O(m)$ packets. To route a packet along a row takes at most $m - 1$ time steps. Once it has arrived at the designated column, it will have to wait for at most $O(m)$ other packets (with high probability). Altogether each packet then needs time $O(m)$ to arrive at its destination.