The Web as a graph

measurements, models, and methods

J. Kleinberg, R. Kumar, P. Raghavan,
S. Rajagopalan, A. Tomkins; 1999

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Seminar of Distributed Computing WS 03/04

Overview

- Introduction
- Algorithms
- Measurements
- Model
- Discussion

1. Introduction

- The Web graph is a directed graph of nodes (pages) and directed edges (hyperlinks)
- Several 100 million nodes (grows exponentially in time)
- Today: more than two billion nodes
- Average node has 7 hyperlinks

Reasons to study Web graph

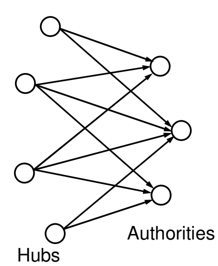
- Improve Web search algorithms
- Topic classification
- Topic enumeration
- Growth of the Web and *behavior of users* is becoming a serious commercial interest

2. Algorithms

- *HITS algorithm* searches for high-quality pages on a topic query
- *Topic enumeration* algorithm enumerates all topics (communities) of the Web graph

Terminology

- *Authoritative* pages are focused on a particular topic
- *Hub* pages contain links to relevant pages on the topic

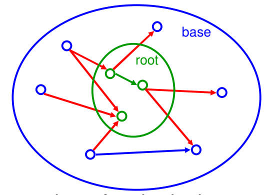


The HITS algorithm

- Hypertext-induced topic selection
- Reveals the most relevant pages on a *search topic*
- Sampling step
- Weight-propagation step

Sampling step

- Construct a subgraph expected to be rich in relevant, authoritative pages
- Keyword query to collect *root set* (~200 pages)
- Expand to *base set* (1000-3000 pages) by including all pages that link to or are linked by a page in the root set



- Base set contains a large number of authoritative pages

Weight-propagation step

- Extract good hubs and authorities from the base set
- Each page p has
  - *authority* weight x_p
  - *hub* weight y_p
- Pages of large hub weights (good hubs) point to pages of large authority weights (good authorities)

Updating weights

- Increase authority weight if page is pointed to by many good hubs:

$$x_p = \sum_{q \to p} y_q$$

- Increase hub weight if page points to many good authorities:

$$y_p = \sum_{p \to q} x_q$$

More mathematical...

- Adjacency matrix A with entries (i,j):
  - 1 if page i links to page j
  - 0 otherwise
- $x = (x_1, x_2, ..., x_n)$
- $y = (y_1, y_2, ..., y_n)$

⇨ new update rules:

$x \leftarrow A^T y$

$y \leftarrow Ax$

...Power iteration

- $x \leftarrow A^T y \leftarrow A^T A x = (A^T A)x$
- $y \leftarrow Ax \leftarrow AA^T y = (AA^T)y$

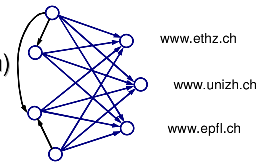
- Multiple iterations → Power iteration
  - k iterations → $(A^T A)^k x$
  - x converges to principal eigenvector of $A^T A$

Conclusion

- Output list contains
  – pages with the largest hub weights
  – pages with the largest authority weights
- After collecting the root set, the algorithm ignores textual content
  Nevertheless it provides *good search results* for a wide range of queries

Topic enumeration

- Enumerates *all* topics (processes *entire* graph)
- *Bipartite core* $C_{i,j}$: contains a complete *bipartite clique* $K_{i,j}$



Bipartite core $C_{4,3}$

- Intuition: Every well represented topic will contain a bipartite core $C_{i,j}$ for some appropriate i and j

⇨ enumerate all bipartite cores for some i and j

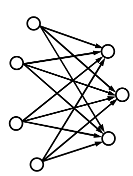
Naive Algorithm

Problems
- Size of search space too large
  10^8 nodes → 10^{40} possibilities
- Requires random access to *edges*
  → large fraction of graph must reside in memory

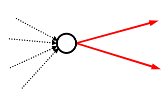
Elimination-generation Algorithm

- Number of sequential *passes* over the graph
- Pass consists of elimination and generation
- During each pass, the algorithm writes a modified version of the graph to the disk
- Alternately sort edges by source and destination (no random access to edges required)

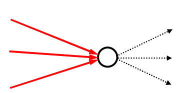
Elimination



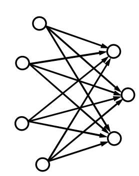
- Consider example $C_{4,3}$
- Edges of nodes with out-degree smaller 3 can be deleted because the node cannot participate on the left side



- Nodes with in-degree smaller 4 cannot participate on the right side



Generation



- *Identify* nodes u that barely qualify for a core
- Either output the core or prove that u doesn't belong to a core, then drop u
- Example: node u with in-degree exactly 4 only belongs to a $C_{4,3}$ if the nodes that point to u have a neighborhood intersection of size at least 3

Observations

- Experiment: over 90% of the cores are not coincidental and correspond to communities with a definite topic focus
- Challenge: How to organize the discovered communities?
- Other interesting subgraphs: webrings, cliques, directed trees

3. Measurements

- Degree distributions
- Number of bipartite cores
- Connectivity of the graph

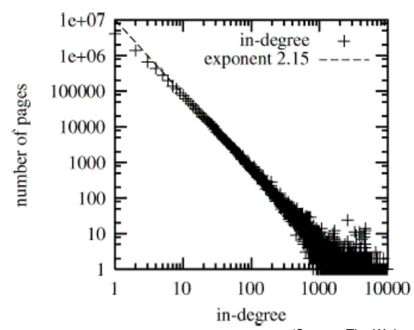
- We will see that traditional random graph models like $G_{n,p}$ don't explain our observations

Degree distributions

- Measurements show that the in- and out-degrees of the nodes are Zipfian distributed
- Zipfian distribution: probability a node has degree i: $P_i \sim 1/i^{\alpha}$, $\alpha \approx 2$
- $G_{n,p}$ has a binomial degree distribution:

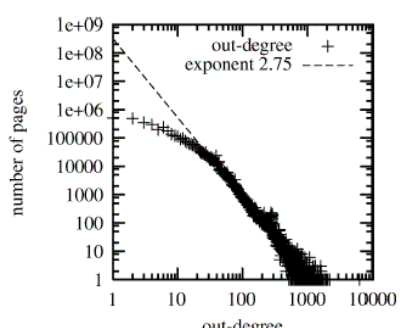
$$P_i = \binom{n}{i} p^i (1-p)^{n-i} \qquad (n = 10^8,\ p = \frac{7.2}{n})$$

In-degree distribution



(Source: The Web graph: an Overview)

Out-degree distribution



(Source: The Web graph: an Overview)

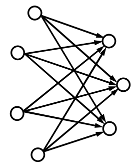
Number of bipartite cores

- Experiment:
  - over 100 million pages
  - i ranging from 3 to 6
  - j ranging from 3 to 9
- Result: number of $C_{i,j}$ in the Web graph
  - over $100'000$ cores
  - $i=3, j=3$: $\approx 40'000$ cores
  - $i=6, j=9$: $\approx 1'000$ cores

Bipartite cores in a random graph

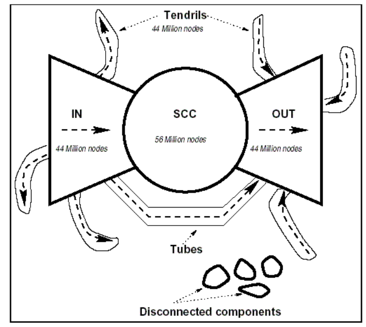
Number of $C_{i,j}$ in $G_{n,p}$, $np = 7.2$ (outdegree):

$$\binom{n}{i}\binom{n}{j}p^{ij} = \binom{n}{i}\binom{n}{j}\left(\frac{7.2}{n}\right)^{ij} \approx \frac{n^{i+j}}{n^{ij}}$$



which is about 0 for $ij > i + j$

Connectivity of the Web



(Source: Graph structure in the web)

Connectivity of the Web

- Bowtie shape
- Strongly connected core (SCC): every page can reach every other by a path (average 20 links)
- IN-pages: can reach the core
- OUT-pages: can be reached by the core
- Scale-free: subgraphs also have the bowtie shape

4. Model

- Reasons for developing a model
- Requirements
- A class of random graph models

Reasons for developing a model

- Model *structural properties* of the graph
  - degrees
  - distribution of $C_{i,j}$
- Predict the behaviour of algorithms on the Web
  - show that an algorithm works well for *problems in the model*, (but would perform bad on worst-case graphs)
- Make predictions about the shape of the Web graph in the *future*

Requirements

- Model should have an easy and natural description
- Capture *aggregate* formation of the graph (not detailed individual behaviour)
- Set of topics *evolve* from the model (no static set required), the Web is *dynamic*
- Reflect the measurements we have seen

A class of random graph models

- Some page creators link to other sites without regard to existing topics
- *Most* page creators link to pages within existing topics of interest
  - find resource list for a topic and include many links from the list in the page → copying links
- *Random copying* as a mechanism to create Zipfian degree distributions

Stochastic processes

- Creation processes C_v and C_e
- Deletion processes D_v and D_e

} discrete time processes

- C_v creates a node with probability $\alpha_c(t)$
- D_v removes a node with probability $\alpha_d(t)$ and also deletes all incident edges
- D_e deletes an edge with probability $\delta(t)$
- Choose probabilities to reflect *growth rates* of the Web, *half-life* of pages, etc.

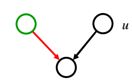
Edge creation process

- Determine a node v and a number k
- With probability β add edges pointing to k uniformly chosen nodes
- With probability $1\text{-}\beta$ copy k edges from a randomly chosen node u
  - If the outdegree of u is more than k, choose a random subset of size k
  - If the outdegree of u is less than k, copy the edges and choose another node u'

A simple model

- New node created at every time step
- No deletions
- Choose u uniformly at random
- β: new edge points to u

- $1\text{-}\beta$: *copy* the out-link from u



Simulation

- Probability a node has indegree i converges to $1/i^{\alpha}$, $\alpha=1/(1\text{-}\beta)$
- Number of cores significantly larger than in a traditional random graph

Challenges

- Study relationship between *copying* and *Zipfian distributions* (applications outside the Web: term frequencies, genome, etc.)
- Study *properties* and evolution of the random graphs generated by the model
- Need *efficient algorithms* to analyze such graphs because copying generates myriads of dependencies