Results for radio broadcast
A Survey

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Theorem (Broadcast Potpourri)

In a multi-hop radio network, broadcast using a deterministic algorithm:

- requires processor ids
- is $\Omega(n)$ for a family of networks
- is $\Omega(D \log(n))$ for another family

There exists a randomized algorithm achieving broadcast with constant probability in $O(D \log N + \log^2 N)$ rounds.
Multi-hop radio network

- Network: undirected graph $G = (V, E)$
- Node: $v \in V$, Turing machine
- Nodes transmit or receive
- $\{v, v'\} \in E$ can communicate
- Message received if *exactly* one neighbour transmits
- Otherwise hear noise
- Synchronous rounds
Finite set of nodes
Start: source has message
Protocol: run locally by nodes
End: all nodes have message
Nodes don’t know topology
Problem: no collision detection
Assume no node ids. Lower bound for a deterministic algo?

<table>
<thead>
<tr>
<th>Network</th>
<th>$\textit{rounds} \geq$</th>
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</thead>
<tbody>
<tr>
<td><img src="image1" alt="Network Diagram" /></td>
<td>1</td>
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<tr>
<td><img src="image2" alt="Network Diagram" /></td>
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<tr>
<td><img src="image3" alt="Network Diagram" /></td>
<td>$n$</td>
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<tr>
<td><img src="image4" alt="Network Diagram" /></td>
<td>$n$</td>
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</tbody>
</table>

Impossible
A family of networks

- Family $C_n$: Source, sink, $n$ layer nodes
- Source connected to all layer nodes
- Some layer nodes connected to sink
- Nodes have ids
- Have to ensure a golden node transmits
- Broadcasting in $C_n$ can be reduced to winning the $n^{th}$ hitting game

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The Hitting Game

Gold = {6, 9, 15}
You win!

Lamps = Blue ∪ Gold

Blue ∩ Gold = ∅

Aim: Find a gold lamp

Initially all lamps off

Move: $M_i \subseteq Lamps$

If $|M_i \cap Gold| = 1$, player wins

If $|M_i \cap Blue| = 1$, switch on

Else “Try Again!”
Adversary Procedure

- For each player strategy, define \textit{Gold} to foil it
- Return \textit{Try Again!} as often as possible
- \textit{Oblivious strategy}: does not depend on referee’s answers
- Sufficient to consider oblivious strategies

\begin{VerbatimBox}{riseright}{left}
\textbf{Find Set. Input: set of moves } M_i \\
Gold := Lamps \\
\textbf{while} winning move } M_i \text{ exists \textbf{do}} \\
\hspace{1em} move } Gold \cap M_i \text{ to } Blue \\
\hspace{1em} \textbf{while} non-singleton blue move } M_j \text{ exists \textbf{do}} \\
\hspace{2em} pick } \ell \in Gold \cap M_j \\
\hspace{2em} move } \ell \text{ to } Blue \\
\hspace{1em} \textbf{end while} \\
\textbf{end while} \\
output } Gold
\end{VerbatimBox}
Adversary Procedure

**Find Set**($M_i$)

$Gold := Lamps$

**while** winning $M_i$ exists

**do**

move $Gold \cap M_i$ to $Blue$

**while** blue $M_j$ exists and $|M_j| > 1$

**do**

pick $\ell \in Gold \cap M_j$

move $\ell$ to $Blue$

**end while**

**end while**

output $Gold$

**Lemma 1**

If the procedure returns the set $Gold$,

- $|M_i \cap Gold| \neq 1$
- $M_i$ is a blue move iff $M_i$ is a singleton

**Lemma 2**

If $|\{M_i\}| \leq \frac{n}{2}$ then the output set $Gold \neq \emptyset$. 
An \( \Omega(n) \) lower bound

- Successful broadcast reduced to winning hitting game
- Rounds for broadcast and steps for winning strategy differ by constant factor of \( \frac{1}{4} \)
- \( n^{th} \) hitting game cannot be won in less than \( \frac{n}{2} \) steps

**Theorem (Lower Bound for \( C_n \))**

There exists no deterministic broadcast protocol which terminates in less than \( \frac{n}{8} \) rounds for any network in \( C_n \).
Towards a better lower bound

- A set $S$ of $k$ disconnected nodes
- In each round $i$, compare transmitting and silent nodes
- Retain the majority in $S$

Lemma: Properties of $S$

After $1 \leq r \leq \log\left(\frac{k}{2}\right)$ rounds

- $|S| \geq 2$
- If a pair $v_1, v_2 \in S$, they have both transmitted together or both remained silent in each round.
Construct a graph $B^D_n$

Diameter $D \leq \frac{n}{2}$

Layers with $\frac{n}{D}$ disconnected nodes

Pick a collision pair from the lemma

Connect to all nodes in next layer

Other nodes connect to fewer nodes in next layer

$D$ such layers
An improved lower bound

- Bruschi and Del Pinto, Distributed Computing 1997
- Independently, Chlebus et. al, SODA 2000
- Given protocol and network, find collision pairs
- Message reaches layer $i$ from $i + 1$ if exactly one node from collision pairs is used
- $n(1 - i/D)$ candidates for collision pairs

**Theorem (Lower Bound for $B^D_n$)**

Given a deterministic broadcast protocol and $n, D \leq \frac{n}{2}$, there exists a network $B^D_n$ which requires $\Omega(D \log(n))$ rounds for broadcast.
First lower bound: $\Omega(n)$
Best known lower bound: $\Omega(D \log(n))$
First distributed algorithm: Diks et al, ESA 1999
First sub-quadratic algo: $O(n^{11/6})$, Chlebus et al, SODA 2000
Non constructive upper bound: $O(n \log^2(n))$, Chrobak, Gasieniec and Rytter, FOCS 2000
Constructive upper bound: Indyk, SODA 2002
A Randomized Protocol

- Does not require node ids
- Input: upper bound on nodes in network and max degree
- Matches lower bound on some network families

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Results for radio broadcast
### Basic Procedure

#### Decay\((k, \text{msg})\)

- \(\text{coin} := \text{heads}\)
- \(\text{steps} := 0\)
- **while** \(\text{coin} = \text{heads}\) and \(\text{steps} \leq k\) **do**
  - send \(\text{msg}\) to neighbours
  - flip \(\text{coin}\)
  - increment \(\text{steps}\)
- **end while**

### Theorem (Decay)

*When \(d \geq 2\) neighbours of a node \(v\) execute \text{Decay} starting simultaneously, the probability that \(v\) receives a message by time \(t\), \(P(t, d)\) satisfies:*

- As \(t \rightarrow \infty\), \(P(t, d) \geq \frac{2}{3}\)
- For \(t \geq 2\lceil \log(d) \rceil\), \(P(t, d) > \frac{1}{2}\)
Broadcast Protocol

Broadcast($N, \Delta$)

$k := 2 \lceil \log \Delta \rceil$

$p := \lceil \log(N/\varepsilon) \rceil$

wait till $msg$ arrives

for $p$ phases do

  wait till $(\text{rnd mod } k) = 0$

  Decay($k, msg$)

end for

- $N$: upper bound on nodes
- $\Delta$: upper bound on max degree
- Runs in $p$ phases
- Execution synchronized in phases to satisfy precondition of theorem.
Broadcast Protocol: Simulation

- $N = 6$
- $\Delta = 4$
- $\epsilon = 0.1$
- $k = 2 \lceil \log(4) \rceil = 4$
- $p = \lceil \log(60) \rceil = 6$

<table>
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<th>$v_2$</th>
<th>$v_3$</th>
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<table>
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<td>$H, m$</td>
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Theorem (Correctness: Message Receipt)

If the processors in a network execute Broadcast, starting with a source s, then:

\[ \Pr(\text{all nodes receive } m) \geq 1 - \varepsilon \]

Proof

\[
\Pr(\text{some } v \text{ does not receive } m) = \Pr(\text{some } v \text{ did not receive } m \text{ but its neighbours did}) \\
\leq \sum_{v \neq s} \Pr(v \text{ didn’t receive } m \text{ but its neighbours did}) \\
\leq n \cdot \left( \frac{1}{2} \right)^{\left\lceil \log(N/\varepsilon) \right\rceil} \\
\leq n \cdot \frac{\varepsilon}{N} \\
\leq \varepsilon
\]
Correctness: Termination - Preliminaries

- \( p(\varepsilon) = \Theta(\max(D, \log(N/\varepsilon))) \)
- \( p(\varepsilon) \): a number of phases considering diameter and conflict delays
- \( rcv(v) \): round in which node \( v \) receives \( msg \)
- \( dist(v_1, v_2) \): length of shortest path from \( v_1 \) to \( v_2 \)
- \( travel(m, p) \): distance traveled by \( m \) in \( p \) phases

Theorem (Correctness: Time for Broadcast)

*If the broadcast protocol runs indefinitely, and \( v \) is the last node to receive the message \( m \), then*

\[
Pr(v \text{ receives } m \text{ in } k \cdot p(\varepsilon) \text{ rounds}) > 1 - \varepsilon
\]
Correctness: Termination - Proof

Consider an arbitrary node \( v \) and the probability that it does not receive a message by a certain round.

Proof

\[
\Pr(\text{ } v \text{ receives after } kp(\varepsilon) \text{ rounds}) \\
= \Pr(\text{ } m \text{ travels less than } \text{dist}(v, s)) \\
\leq \Pr(\text{ } m \text{ travels less than } D) \\
= \Pr(\sum_{i=1}^{p(\varepsilon)} \text{travel}(m, 1) < D) \\
\leq \frac{\varepsilon}{N} \\
\Pr(rcv(v) \leq k \cdot p(\varepsilon)) > 1 - \frac{\varepsilon}{N} > 1 - \varepsilon
\]
Randomization “beats” deterministic impossibility result

Expected running time: $O(D \log N + \log^2 N)$

Lower bound: $\Omega(D \log(N/D))$, Kushilevitz and Mansour, SIAM Journal of Computing, 1998

Alternate proof: Liu and Prabhakaran, COCOON, 2002


Broadcast and gossip studied together in recent work
Conclusion

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In a multi-hop radio network, broadcast using a deterministic algorithm:

- requires processor ids
- is $\Omega(n)$ for a family of networks
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There exists a randomized algorithm achieving broadcast with constant probability in $O(D \log N + \log^2 N)$ rounds.
Example: Multi-hop radio network
Reduction of broadcast to hitting games
Lower bound for randomized algorithms
The sensor my friend is blowing in the wind

- Sensor nodes around 100 cubic millimeter large
- Includes temperature, light sensors, bi-directional wireless communication
- Set of nodes: distributed system
- Many potential applications
Reduction to the hitting game

- Reduce broadcast protocol to restricted protocol in which either source or sink is active
- Reduce restricted protocol to abstract protocol where only middle nodes transmit and either source or sink receives
- Middle nodes know when transmission is successful
- Abstract broadcast achieved when a gold node successfully transmits
- Reduce abstract broadcast to hitting game
Consider the blue-gold network family

Yao’s minimax principle: reduce proving randomized lower bound to deterministic lower bound on probabilistic input

Construct probability distribution of inputs

Lower bound of $\Omega(\log m)$

Connect $D$ layers with $N/D$ nodes in each

Some calculations later: $\Omega(D \log(N/D))$